Definitions and Examples for Topology

**Abelian**
A group is *abelian* if \( ab = ba \) for all \( a, b \in G \).

**Bijection**
A *bijection* from \( X \) to \( Y \) is a map \( f : X \rightarrow Y \) which is both one-to-one and onto.

**Brouwer Fixed Point Theorem**
Every continuous \( f : D^2 \rightarrow D^2 \) has a fixed point.

**Cayley Graph**
The *Cayley Graph* of a (finitely generated) group is a graph consisting of one vertex for each group element, and out of each vertex, one directed edge per generator, so that if \( g_1 \cdot a = g_2 \)

\[
\text{generator}
\]

add edge from \( g_1 \) to \( g_2 \).

**Conjugate**
A *conjugate* of a word \( w \) in \( G \) is \( gwg^{-1} \), where \( g \) is an element of \( G \).

**Example**
\[ \mathbb{Z} \times \mathbb{Z} \cong \langle a, b | aba^{-1}b^{-1} \rangle \]
\( \phi : \langle a, b \rangle \rightarrow \mathbb{Z} \times \mathbb{Z} \) by
\[
\phi(a) = (1, 0),
\phi(b) = (0, 1),
\phi(aba^{-1}b^{-1}) = (1, 0) + (0, 1) + (-1, 0) + (0, -1) = (0, 0).
\]

**Constant**
A function \( f : X \rightarrow Y \) is *constant* if there is some \( c \in Y \) so that \( f(x) = c \) for every \( x \in X \).

**Constant loop**
The *constant loop* in a space \( X \) based at \( x_0 \in X \) is
\[
C : (I, dI) \rightarrow (x, x_0)
\]
by \( C(s) = x_0 \).
Continuity
A map $f$ from a topological space $X$ to a topological space $Y$ is continuous if and only if for every set $U \subseteq Y$ that is open in $Y$, the set $f^{-1}(U) \subseteq X$ is open in $X$.

Example
$X = \{a, b, c\}$, $\tau_X = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$
$Y = \{w, z\}$, $\tau_Y = \{\emptyset, Y, \{w\}\}$
Define $f : X \rightarrow Y$ by

\[
\begin{align*}
f(a) &= w \\
f(b) &= z \\
f(c) &= z
\end{align*}
\]

Is $f$ continuous?
The open sets in $Y$ are $\emptyset, Y, \{w\}$.
\[
\begin{align*}
f^{-1}(\emptyset) &= \emptyset \in \tau_X \\
f^{-1}(Y) &= X \in \tau_X \\
f^{-1}(\{w\}) &= \{a\} \in \tau_X
\end{align*}
\]

So $f$ is continuous.

Contractible
A space which is homotopy equivalent to a point is called contractible.

Proposition
$X$ is contractible if and only if $I_x \simeq \text{constant} : X \rightarrow X$.

Proof
$\Rightarrow$ Assume $X$ is contractible. We want to show $I_x \simeq \text{constant}$. Then by definition, $X$ is homotopy equivalent to a point. So there's some point $y$ and maps

\[
\begin{align*}
f : X &\rightarrow \{y\} \\
g : \{y\} &\rightarrow X
\end{align*}
\]

so that

\[
\begin{align*}
g \circ f &\simeq I_x \\
f \circ g &\simeq I_{\{y\}}
\end{align*}
\]

Note, if $x \in X$, then $g \circ f(x) = g(y)$.

So $g \circ f$ maps every point in $X$ to the same point in $X$ (namely $g(y)$). So $g \circ f$ is a constant map. So $I_x \simeq \text{constant}$ map.

$\Leftarrow$ Assume $I_x \simeq \text{constant} : X \rightarrow X$. Then $I_x \simeq C$, where $C$ is a map $C : X \rightarrow X$ and $x_0 \in X$ and $C(x) = x_0$ for all $x \in X$. We want to show $X$ is contractible.
Define $f : X \rightarrow \{x_0\}$ by $f(x) = x_0$ for all $x \in X$.
Define $g : \{x_0\} \rightarrow X$ by $g(x_0) = x_0$.
Now $f \circ g(x_0) = f(x_0) = x_0$.
So $f \circ g = I_{\{x_0\}}$.
Therefore, $f \circ g \simeq I_{\{x_0\}}$.
Further, $g \circ f(x) = g(x_0) = x_0$.
So $g \circ f = C \simeq \tau_x$.
So $X$ is homotopy equivalent to a point; $X$ is contractible.
Covering space

$p : E \rightarrow B$ is a covering space (projection) if $E$ and $B$ are path connected and for all $b \in B$, there exists a path connected neighborhood $U$ of $b$ such that every component of $p^{-1}(U)$ maps homeomorphically (via $p$) onto $U$.

Finite presentation

A finite presentation for a group $G$ is $\langle g_1, g_2, g_3, \ldots, g_n | R_1, R_2, \ldots, R_m \rangle$ so that

1. Every element of $G$ is a finite product of $g_i$’s and then inverses. ($g_i$’s are generators).
2. Each relation $R_j$ is a word in the $g_i$’s (and their inverses) which gives the identity element in $G$.
3. If a reduced word in $g_i$’s is the identity element in $G$, then that reduced word is obtained by reducing some product of conjugates of the relatives.

Fixed point

A fixed point of a function $f : X \rightarrow x$ is an element $x \in X$ so that $f(x) = X$.

Example

$f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = 2x$.

$0$ is the only fixed point.

Free product

The free product of two finitely presented groups $G$ and $H$ is $G \ast H = \langle$generators for $G$, generator for $H$ relations for $G$, relations for $H \rangle$.

Example

$$Z \simeq \langle || \rangle$$

$$= \langle a \rangle$$

$$= \langle b \rangle$$

$$Z \ast Z \simeq \langle a, b \rangle = F_2$$

$$Z \times Z \simeq (a, b) | aba^{-1}b^{-1}$$
**Function**
If \( f : X \to Y \) is a function and \( U \subseteq Y \), then \( f^{-1}(U) = \{ x \in X : f(x) \in U \} \).

**Example**
\( f : \mathbb{R} \to \mathbb{R} \) is \( f(x) = x^2 \)

\[
f^{-1}\{\sqrt{2}\} = \{-\sqrt{2}, \sqrt{2}\} \\
f^{-1}\{-\sqrt{2}\} = \emptyset \\
f^{-1}\{(1, 2)\} = (-\sqrt{2}, -1) \cup (1, \sqrt{2})
\]

**Group**
A **Group** consists of a set \( G \) and an operation \( \cdot \) defined on that set \( (\cdot : G \times G \to G) \) so that
1. Identity: There is an element \( e \in G \) so that for every \( g \in G \)
   \[
e \cdot g = g \quad \text{and} \quad g \cdot e = g
   \]
2. Inverses: For all \( g \in G \), there is \( h \in G \) so that
   \[
g \cdot h = e \quad \text{and} \quad h \cdot g = e
   \]
3. Associativity: If \( g, h, j \in G \), then
   \[
   (g \cdot h) \cdot j = g \cdot (h \cdot j)
   \]

**Homeomorphic**
Two topological spaces \( X \) and \( Y \) are **homeomorphic** if there exists a homeomorphism \( F : X \to Y \).

**Homeomorphism**
A bijection \( f : X \to Y \) such that \( f \) and \( f^{-1} \) are both continuous is called a **homeomorphism**.

**Homeotopic**
Two maps \( f, g : X \to Y \) are **homeotopic** if there exists a map
\[
F : X \times I \to Y
\]
such that for each \( x \in X \)
\[
F(x, 0) = f(x) \quad \text{and} \quad F(x, 1) = g(x)
\]
We say \( f \) is homotopic to \( g \), write \( f \simeq g \), and \( F \) is a **homotopy** between \( f \) and \( g \).
Homomorphic
Two groups \((G, \ast_G)\) and \((H, \ast_H)\) are **homomorphic** if there exists a function \(f : G \rightarrow H\) so that, for every \(g_1, g_2 \in G\),
\[
f(g_1 \ast_G g_2) = f(g_1) \ast_H f(g_2)
\]

Homotopic
Two paths \(f, g : I \rightarrow X\) are **homotopic** if and only if there exists \(F : I \times I \rightarrow X\) so that
\[
F(x, 0) = f(x) \\
F(x, 1) = g(x)
\]

Example
Define \(f, g : \mathbb{R} \rightarrow \mathbb{R}\) by \(f(x) = 2\), \(g(x) = 5\), for every \(x \in \mathbb{R}\).

Show \(f\) and \(g\) are homotopic. We need a map \(F : \mathbb{R} \times I \rightarrow \mathbb{R}\) so that
\[
F(x, 0) = f(x) = 2 \\
F(x, 1) = g(x) = 5
\]
Define \(F : \mathbb{R} \times I \rightarrow \mathbb{R}\) by
\[
F(x, t) = 3t + 2
\]
So
\[
F(x, 0) = 3(0) + 2 = 2 = f(x) \\
F(x, 1) = 3(1) + 2 = 5 = g(x)
\]

Homotopic relative to \(A\)
Two maps \(f, g : (X, A) \rightarrow (Y, B)\) are **homotopic relative to \(A\)**, written \(f \simeq \text{grelA}\), if there exists a homotopy \(F : (X \times I, A \times I) \rightarrow (Y, B)\) so that
\[
F(x, 0) = f(x), \quad F(x, 1) = g(x) \text{ for all } x \in X \text{ and for all } x \in A \\
F(x, t) = f(x) \text{ for every } t \in I.
\]

Homotopy class
The **homotopy class** of a loop \(\alpha\) in \(X\) based at \(x_0 \in X\) is
\[
[\alpha] = \{ \beta : (I, dI) \rightarrow (x, x_0) : \alpha \simeq \beta \text{ rel } dI \}
\]
Homotopy equivalent
Two spaces $X$ and $Y$ are homotopy equivalent if there exists maps $f : X \to Y$ and $g : Y \to X$ so that

\[
g \circ f \simeq I_x \\
f \circ g \simeq I_Y
\]

Identity map
Let $X$ be a topology space. Then $I_x$ is the function $I_x : X \to X$ defined by $I_x(x) = X$

This is called the identity map on $X$.

Inclusion map
If $A \subseteq X$, then the inclusion map $i : A \to X$ is $i(a) = a$ for all $a \in A$.

Induced map
Suppose $f : (X, x_0) \to (Y, y_0)$ and suppose $\alpha : (I, dI) \to (X, x_0)$ is a loop in $X$. Then $f \circ \alpha : (I, dI) \to (Y, y_0)$ is a loop in $Y$. This gives a map $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, $f_*([\alpha]) = [f \circ \alpha]$

$f_*$ as above is called the induced map on $\pi_1$.

Proposition: $f_*$ is a homomorphism
Proof: We want to show $f_*([\alpha]) \cdot f_*([\beta]) = f_*([\alpha] \cdot [\beta])$ for all $[\alpha][\beta] \in \pi_1(X, x_0)$. Let $[\alpha][\beta] \in \pi_1(X, x_0)$.

Then, $f_*([\alpha]) \cdot f_*([\beta]) = [f \circ \alpha] \cdot [f \circ \beta] = [f \circ \alpha \cdot f \circ \beta]$ where

\[
(f \circ \alpha \cdot f \circ \beta)(s) = \begin{cases} f \circ \alpha(2s) & 0 \leq s \leq 1/2 \\ f \circ \beta(2s - 1) & 1/2 \leq s \leq 1 \end{cases}
\]

$f_*([\alpha \cdot \beta]) = f_*([\alpha] \cdot [\beta]) = [f \circ (\alpha \cdot \beta)]$

where,

\[
(f \circ (\alpha \cdot \beta))(s) = \begin{cases} \alpha(2s) & 0 \leq s \leq 1/2 \\ \beta(2s - 1) & 1/2 \leq s \leq 1 \end{cases}
\]

These are the same.
**Isomorphic**

Two groups \((G, \ast_G), (H, \ast_H)\) are isomorphic, if there exists \(f : G \rightarrow H\) which is a bijective homomorphism. \(G \cong H\).

**Proposition**

\[(G, \cdot_G) \cong (H, \cdot_H)\]

Define \(f : G \rightarrow H\) by

\[f(e) = 1\]

\(f\) is clearly a bijection. To see \(f\) if a homomorphism, choose \(g_1, g_2 \in G\). Then \(g_1 = e = g_2\). So,

\[f(g_1 \cdot_G g_2) = f(e \cdot_G e)\]
\[= f(e)\]
\[= 1\]

\[f(g_1 \cdot_H g_2) = f(e \cdot_H e)\]
\[= 1 \cdot_H 1\]
\[= 1\]

So \(G \cong H\).

**Knot complement**

Let \(K\) be a knot: \(K = f(s^1)\). Then \(X = S^3 - K\) is the knot complement of \(K\).

**Loop**

A loop in \(X\) is a path \(f : I \rightarrow X\) so that \(f(0) = f(1)\).

**Map of Pairs**

A map of pairs is \(f : (x, A) \rightarrow (Y, B)\) which means

1. \(f : X \rightarrow Y\)
2. \(f(A) \subseteq B\)

**Nullhomotopic**

A function \(f : X \rightarrow Y\) is nullhomotopic if it is homotopic to a constant map.

**One-to-One**

A function \(f : X \rightarrow Y\) is one-to-one if and only if, for all \(a, b \in X\), \(f(a) = f(b) \rightarrow a = b\).
Onto
A function \( f : X \rightarrow Y \) is \emph{onto} if and only if, for all \( a \in X \), there exists \( b \in X \), \( f(b) = a \).

Pair
A \emph{pair} of topological spaces \( (x, A) \) is a topological space \( X \) with \( A \subseteq X \).

Path
A map \( f : I \rightarrow X \) so that \( f(0) = x_0 \) and \( f(1) = x_1 \) is called a \emph{path} in \( X \) from \( x_0 \) to \( x_1 \). (Note: Path is a map, not the image of a map.)

Path Connected
A space \( X \) is \emph{path connected} if for every \( x_0, x_1 \in X \), there exists a path \( f : I \rightarrow X \) so that \( f(0) = x_0 \) and \( f(1) = x_1 \).

Path Homotopic
Two paths \( f, g : I \rightarrow X \) are \emph{path homotopic} or \emph{homotopic rel endpoints}, or \emph{homotopic rel boundary}. If they have the same initial point \( x_0 \) and the same final point \( x \), and there’s a homotopy
\[
F : I \times I \rightarrow X
\]
so that
\[
F(s, 0) = f(s), F(s, 1) = g(s) \text{ for all } s \in I
\]
\[
F(0, t) = x_0, F(1, t) = x, \text{ for all } t \in I.
\]

Reduced word
A \emph{reduced word} is a word in which you do not have a letter next to its inverse.

Example
\( aboa^{-1}b \) Not reduced
\( abb \) is reduced.
This group above is the \underline{free} group of rank 2, written \( F_2 \).

Retraction of \( X \) onto \( A \)
If \( A \subseteq X \), then a \emph{retraction of \( X \) onto \( A \)} is a continuous function \( r : X \rightarrow A \) with \( r|_A = I_A \) (alternately \( r \circ i = I_A \))  
"Elements of \( A \) are fixed by the function \( r \)."

Example
Let \( A = 0, X = \mathbb{R} \), then \( A \subseteq X \).
Define \( r : X \rightarrow A \)
\[
\mathbb{R} \rightarrow \{0\} \text{ by } r(x) = 0.
\]
Then \( r(0) = 0 \), so \( r \) fixes elements of \( A \). So \( r \) is a retraction.
Simply connected
A space $X$ is called simply connected if it is path connected and $\pi_1(X, x_0) \simeq \{1\}$, for every $x_0 \in X$.

**Theorem** $\pi_1(s^1, x_0) \simeq (\mathbb{Z}, +)$

**Proof**
We want do define a map

$$ f : \pi_1(s^1, (1, 0)) \rightarrow \mathbb{Z} $$

which is an isomorphism.

To define $f$, let $[\alpha] \in \pi_1(s^1, (1, 0))$

Then $\alpha : (I, dI) \rightarrow (s^1, (1, 0))$. Using the Unique Path Lifting lemma there exists $\alpha(0) = 0$.

Define

$$ n_\alpha = \tilde{\alpha}(1) $$

**Claim:** Note that $n_\alpha$ is an integer.

**Proof Claim:** we know $\pi \circ \alpha = \alpha$

$$ \Rightarrow \pi(\tilde{\alpha}(1)) = \alpha(1) $$

$$ \Rightarrow \pi(\tilde{\alpha}(1)) = (1, 0) $$

The function $\pi$ is defined by

$$ \pi(x) = (\cos 2\pi x, \sin 2\pi x) = (1, 0) $$

when $\cos 2\pi x = 1$ and $\sin 2\pi x = 0$.

This happens only when $x \in \mathbb{Z}$.

So

$$ \pi(\tilde{\alpha}(1)) = (1, 0) \Rightarrow n_\alpha = \tilde{\alpha}(1) \in \mathbb{Z} $$

For $[\alpha] \in \pi_1(s^1, (1, 0))$, define $f[\alpha] = n_\alpha = \alpha(1)$.

Need to check $f$ is 1.well-defined 2. 1-1 3.onto 4.homomorphism.

1.**Well-defined:** Show $[\alpha] = [\beta]$, then $f[\alpha] = f[\beta]$.

Assume $[\alpha] = [\beta]$. In other words, assume $\alpha, \beta : (I, dI) \rightarrow (s^1, (1, 0))$ with $\alpha \simeq \beta$ rel $dI$.

[want to show $n_\alpha = n_\beta$].

Since $\alpha \simeq \beta$ rel $dI$, there exists homotopy $F : (I \times I, dI \times I) \rightarrow (s^1, (1, 0))$.

So that

$$ F(s, 0) = \alpha(s) $$

$$ F(s, 1) = \beta(s) $$

$$ F(0, t) = (1, 0) $$

$$ F(1, t) = (1, 0) $$

For each $t$, let $a_t(s) = F(s, t)$. This is a loop in $s^1$ based at (1,0). Define function $n : \{\text{loops in } s^1\} \rightarrow \mathbb{Z}$ by $n(\gamma) = nr$. Since this is a continuous map into a discrete set, it must be constant. So $n_{\alpha_t} = n_{\beta_t}$ for every $t_1, t_2$. So $n_\alpha = n_\beta$ and $f$ is well defined.

2. **1-1** Suppose $f([\alpha]) = f([\beta])$. We want to show $[\alpha] = [\beta]$.

So $n_\alpha = n_\beta$. We want to define a homotopy $F : (I \times I, dI) \rightarrow (s^1, (1, 0))$. So that

$$ f(s, 0) = \alpha(s) $$

$$ f(s, 1) = \beta(s) $$

$$ F(0, 0) = (1, 0) $$

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Since $\alpha, \beta : (I, dI) \rightarrow (s^1, (1, 0))$, the unique path lifting lemma says there exists lifts $\tilde{\alpha}, \tilde{\beta} : (I, dI) \rightarrow \mathbb{R}$ with $\tilde{\alpha} = 0 = \tilde{\beta}(0)$.

Since $f(\alpha) = f([\beta])$, we know $\tilde{\alpha}(1) = \tilde{\beta}(1)$.

Define homotopy $\tilde{F} : I \times I \rightarrow \mathbb{R}$ by
\[
\tilde{F}(s, t) = (1 - t)\alpha(s) + t(p) \in \mathbb{R}
\]
\[
\tilde{F}(s, 0) = \alpha(s)
\]
\[
\tilde{F}(s, 1) = \beta(s)
\]
\[
\tilde{F}(0, t) = (1 - t)\alpha(1) - t\beta(1) = (-t)\alpha(1) + t\alpha(1) = \alpha(1)
\]
\[
\tilde{F}(0, t) = (1 - t)(0 + t(0) = 0
\]

So $\tilde{\alpha} \simeq \tilde{\beta}$ rel $dI$.

Now define $F$ by $F = \pi \circ \tilde{F}$. Then $F : I \times I \rightarrow s^1$, and
\[
f(s, 0) = \pi \circ \tilde{F}(s, 0)
\]
\[
f(s, 0) = \pi \circ \tilde{\alpha}(s)
\]
\[
f(s, 0) = \alpha(s)
\]
\[
f(s, 1) = \pi \circ \tilde{F}(s, 1)
\]
\[
f(s, 1) = \pi \circ \tilde{\beta}(s)
\]
\[
f(s, 1) = \beta(s)
\]
\[
f(0, t) = \pi \circ \tilde{F}(0, t)
\]
\[
f(0, t) = \pi(0)
\]
\[
f(0, t) = (1, 0)
\]
\[
f(1, t) = \pi \circ \tilde{F}(1, t)
\]
\[
f(1, t) = \pi(\tilde{\alpha}(1))
\]
\[
f(1, t) = (1, 0)
\]

3. onto: Let $m \in \mathbb{Z}$. We want to find $[\alpha]$ such that $f([\alpha]) = m$.

Define $\alpha_m : (I, dI) \rightarrow (s^1, (1, 0))$ by
\[
\alpha_m(s) = (\cos(2\pi sm), \sin(2\pi sm))
\]

Then $\alpha(1) = m$. So $f([\alpha]) = n_\alpha = \alpha(1) = m$.

4. homomorphism: Let $[\alpha], [\beta] \in \pi_1(s^1, (1, 0))$. We want to show
\[ f([\alpha] \cdot [\beta]) = f([\alpha]) + f([\beta]) \]

Then \( f([\alpha]) = n \) and \( f([\beta]) = m \). So \( \alpha \) is homotopic to the standard loop wrapping around the circle \( n \) times:

\[
\begin{align*}
\alpha & \simeq \alpha_n \text{ rel } \partial I \\
\beta & \simeq \alpha_m \text{ rel } \partial I
\end{align*}
\]

\[
\begin{align*}
 f([\alpha] \cdot [\beta]) & = f([\alpha_n] \cdot [\alpha_m]) \\
 & = f([\alpha \cdot \alpha_m]) \\
 & = f([\alpha(n + m)]) \\
 & = n + m \\
 & = f([\alpha_n]) + f([\alpha_m]) \\
 & = f([\alpha]) + f([\beta])
\end{align*}
\]

**Strong deformation retraction**

If \( A \subseteq X \), and \( r : X \to A \) is a retraction, then \( r \) is a *strong deformation retraction* if \( I_x \simeq r \text{ rel } A \).

**Example**

![Diagram](image)

There is an SDR from \( X \) to \( A \).

**Word**

A *word* in the symbols \( a, b \) is a finite sequence of elements chosen from \( \{a, a^{-1}, b, b^{-1}\} \).

**Example**

\( aaba^{-1}bbaab^{-1} \) is a word.