**Convolution Representation of Linear Time-Invariant Continuous-Time Systems**

**Impulse Response**

Recall the definition of an impulse function:

\[ \delta(t) = 0 \quad , \quad t \neq 0 \]

\[ \int \delta(\lambda) d\lambda = 1 \ . \]

The impulse response is,

\[ h(t) = \delta(t) \ . \]

For a causal system, \( h(t) = 0 \) for all \( t < 0 \).

![System impulse response diagram](image)

If \( x(t) \) is a causal function ( \( x(t) = 0 \) for all \( t < 0 \) ), by the sifting property,

\[ x(t) = \int_{0}^{\infty} x(\lambda) \delta(t-\lambda) \ d\lambda \ . \]

The output is,

\[ y(t) = x(t) * h(t) \quad , \quad t \geq 0 \]

\[ = \int_{0}^{\infty} x(\lambda) h(t-\lambda) \ d\lambda \]

which is the convolution representation.
Therefore,

\[ x(t) \ast v(t) = \begin{cases} 0, & t < 0 \\ \int_0^t x(\lambda)v(t-\lambda)d\lambda, & t > 0 \end{cases} \]

The integral exists if \( x(t) \) and \( v(t) \) are integrable. That is,

\[ \int_0^t |x(\lambda)|d\lambda < \infty \]

\[ \int_0^t |v(\lambda)|d\lambda < \infty, \quad t > 0 \]
The following steps are recommended when performing convolutions of two functions $x(t)$ and $v(t)$.

Step 1. Graph $x(\lambda)$ and $v(-\lambda)$ (or vise-versa) as functions of $\lambda$. That is, fold $v(\lambda)$ about the vertical axis.

Step 2. Upon folding $v(\lambda)$, denote the point where $\lambda = 0$ as $t$. All other points of $v(-\lambda)$ are relative to the reference point $t$. Remember, we're now in the $\lambda$-axis.

Step 3. Define an interval for which the product $x(\lambda)v(t-\lambda)$ has the same analytical form (e.g. from $t = 0$ to $t = 1$).

Step 4. Integrate the product $x(\lambda)v(t-\lambda)$ as a function of $\lambda$ over the integral defined in Step 3.

Step 5. Repeat Steps 3 and 4 as many times as necessary until $x(\lambda)v(t)$ is computed for all $t > 0$.

**EXAMPLE**

Convolve the following functions:

$$x(t) = \Pi \left( \frac{t - 1}{2} \right)$$

$$v(t) = 2\Pi \left( t - \frac{1}{2} \right)$$
Choose to flip \( v(t) \) about the y-axis. This will then be the convolution at \( t = 0 \).

The first region of integration is \( t < 0 \) for which no overlapping area exists. We construct a table as follows:

<table>
<thead>
<tr>
<th>Time shift, ( t )</th>
<th>( \lambda ) lower limit of integration</th>
<th>( \lambda ) upper limit of integration</th>
<th>Area of Overlap (Integral)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t &lt; 0 )</td>
<td>---</td>
<td>---</td>
<td>0</td>
</tr>
</tbody>
</table>
For ease of viewing the "overlap" intervals, the two functions are drawn on the same axes:

The function $v(t - \lambda)$ slides over the function $x(\lambda)$. The second region of overlap and function to be integrated is $0 \leq t < 1$ with the lower limit of the overlapping area determined by $x(\lambda)$ and the upper limit $t$ of the area is determined by $v(t - \lambda)$.

<table>
<thead>
<tr>
<th>Time shift, $t$</th>
<th>$\lambda$ lower limit of integration</th>
<th>$\lambda$ upper limit of integration</th>
<th>Area of Overlap (Integral)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t &lt; 0$</td>
<td>---</td>
<td>---</td>
<td>0</td>
</tr>
<tr>
<td>$0 \leq t &lt; 1$</td>
<td>$0$</td>
<td>$t$</td>
<td>$\int_{\lambda_{lower}}^{\lambda_{upper}} (2)(1) d\lambda$</td>
</tr>
</tbody>
</table>
The third region of overlap is $1 \leq t < 2$ as shown below. There is complete overlap of the two areas. Therefore, the integration limit of the overlap area are from $t - 1$ to $t$ defined by $v(t - \lambda)$ in the region $1 \leq t < 2$.

![Graph showing continuous convolution](image)

<table>
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<tr>
<th>Time shift, $t$</th>
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<th>$\lambda$ upper limit of integration</th>
<th>Area of Overlap (Integral)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t &lt; 0$</td>
<td>---</td>
<td>---</td>
<td>0</td>
</tr>
<tr>
<td>$0 \leq t &lt; 1$</td>
<td>0</td>
<td>$t$</td>
<td>$\int_{\lambda_{\text{lower}}}^{\lambda_{\text{upper}}} (2)(1) d\lambda$</td>
</tr>
<tr>
<td>$1 \leq t &lt; 2$</td>
<td>$t - 1$</td>
<td>$t$</td>
<td>$\int_{\lambda_{\text{lower}}}^{\lambda_{\text{upper}}} (2)(1) d\lambda$</td>
</tr>
</tbody>
</table>
The fourth region of overlap is $2 \leq t < 3$ over which the upper limit of integration is 2 as defined by $x(\lambda)$ and the lower limit is $t - 1$ as defined by $v(t - \lambda)$.

<table>
<thead>
<tr>
<th>Time shift, $t$</th>
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<th>$\lambda$ upper limit of integration</th>
<th>Area of Overlap (Integral)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t &lt; 0$</td>
<td>---</td>
<td>---</td>
<td>$0$</td>
</tr>
<tr>
<td>$0 \leq t &lt; 1$</td>
<td>0</td>
<td>$t$</td>
<td>$\int_{\lambda_{lower}}^{\lambda_{upper}} (2)(1) d\lambda$</td>
</tr>
<tr>
<td>$1 \leq t &lt; 2$</td>
<td>$t - 1$</td>
<td>$t$</td>
<td>$\int_{\lambda_{lower}}^{\lambda_{upper}} (2)(1) d\lambda$</td>
</tr>
<tr>
<td>$2 \leq t &lt; 3$</td>
<td>$t - 1$</td>
<td>2</td>
<td>$\int_{\lambda_{lower}}^{\lambda_{upper}} (2)(1) d\lambda$</td>
</tr>
</tbody>
</table>
The last region is trivial in that there is no overlap between the two functions. The resultant area of overlap is 0.

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Time shift, } t & \lambda \text{ lower limit of integration} & \lambda \text{ upper limit of integration} & \text{Area of Overlap (Integral)} \\
\hline
\text{Region of Overlap} & \lambda_{\text{lower}} & \lambda_{\text{upper}} & \\
\hline
\text{ } & \text{ } & \text{ } & 0 \\
\text{ } & \text{ } & \text{ } & \int_{\lambda_{\text{lower}}}^{\lambda_{\text{upper}}} (2)(1) \, d\lambda \\
\text{ } & \text{ } & \text{ } & \int_{\lambda_{\text{lower}}}^{\lambda_{\text{upper}}} (2)(1) \, d\lambda \\
\text{ } & \text{ } & \text{ } & \int_{\lambda_{\text{lower}}}^{\lambda_{\text{upper}}} (2)(1) \, d\lambda \\
\text{ } & \text{ } & \text{ } & \int_{\lambda_{\text{lower}}}^{\lambda_{\text{upper}}} (2)(1) \, d\lambda \\
\text{ } & \text{ } & \text{ } & \int_{\lambda_{\text{lower}}}^{\lambda_{\text{upper}}} (2)(1) \, d\lambda \\
\hline
\text{ } & \text{ } & \text{ } & 0 \\
\hline
\end{array}
\]
The result of the convolution is:

\[
x(t) * v(t) = \begin{cases} 
0 & , t < 0 \\
2t & , 0 \leq t < 1 \\
2 & , 1 \leq t < 2 \\
-2t + 6 & , 2 \leq t < 3 \\
0 & , t \geq 3 
\end{cases}
\]

Result of the convolution.