Mathematics Framework & Pedagogy
[Source: A Guide to Teaching & Learning of 'O' Level Mathematics 2007 (Curriculum Planning and Development Division, Ministry of Education (Singapore), 2006)]

THE FRAMEWORK

Figure 1: Framework of School Mathematics Curriculum

Figure 1 shows the underlying principles of an effective mathematics programme that is applicable to all levels, from the primary to advanced levels. It sets the direction for the teaching, learning, and assessment of mathematics.

Mathematical problem solving is central to mathematics learning. It involves the acquisition and application of mathematics concepts and skills in a wide range of situations, including non-routine, open-ended and real-world problems. The development of mathematical problem solving ability is dependent on five inter-related components, namely, Concepts, Skills, Processes, Attitudes and Metacognition.

Concepts
Mathematical concepts cover numerical, algebraic, geometrical, statistical, probabilistic, and analytical concepts. Students should develop and explore the mathematics ideas in depth, and see that mathematics is an integrated whole, not merely isolated pieces of knowledge. They should be given a variety of learning experiences to help them develop a deep understanding of mathematical concepts, and to make sense of various mathematical ideas, as well as their connections and applications, in order to participate
actively in learning mathematics and to become more confident in exploring and applying mathematics. The use of manipulatives (concrete materials), practical work, and use of technological aids should be part of the learning experiences of the students.

Skills
Mathematical skills include procedural skills for numerical calculation, algebraic manipulation, spatial visualization, data analysis, measurement, use of mathematical tools, and estimation. The development of skills proficiencies in students is essential in the learning and application of mathematics. Although students should become competent in the various mathematical skills, over-emphasizing procedural skills without understanding mathematical principles should be avoided. Skill proficiencies include the ability to use technology confidently, where appropriate, for exploration and problem solving. It is important also to incorporate the use of thinking skills and heuristics in the process of the development of skills proficiencies.

Processes
Mathematical processes refer to the knowledge skills (or process skills) involved in the process of acquiring and applying mathematical knowledge. This includes reasoning, communication and connections, thinking skills and heuristics, and application and modeling.

Reasoning, communication and connections
Mathematical reasoning refers to the ability to analyze mathematical situations and construct logical arguments. It is a habit of mind that can be developed through the application of mathematics in different situations and contexts.

Communication refers to the ability to use mathematical language to express mathematical ideas and arguments precisely, concisely and logically. It helps students develop their own understanding of mathematics and sharpen their mathematical thinking.

Connections refer to the ability to see and make linkages among mathematical ideas, between mathematics and other subjects, and between mathematics and everyday life. This helps students make sense of what they learn in mathematics.

Mathematical reasoning, communication and connections should pervade all levels of mathematics learning, from primary levels to the advanced-levels.

Thinking skills and heuristics
Students should use various thinking skills and heuristics to help them solve mathematical problems. Thinking skills are skills that can be used in a thinking process, such as classifying, comparing, sequencing, analyzing parts and wholes, identifying patterns and relationships, induction, deduction and spatial visualization. Some examples of heuristics are listed below in four categories according to how they are used:

- To give a representation, e.g.
  - Draw a diagram, make a list, use equations
○ To make a calculated guess, e.g.  
  Guess and check, look for patterns, make suppositions  
○ To go through the process, e.g.  
  Act it out, work backwards, before-after  
○ To change the problem, e.g.  
  Restate the problem, simplify the problem, solve part of the problem  

**Applications and modeling**  
Applications and modeling play a vital role in the development of mathematical understanding and competencies. It is important that students apply mathematical problem solving skills and reasoning skills to tackle a variety of problems, including real-world problems.  

**Mathematical modeling** is the process of formulating and improving a mathematical model to represent and solve real-world problems. Through mathematical modeling, students learn to use a variety of representations of data, and to select and apply appropriate mathematical methods and tools in solving real-world problems. The opportunity to deal with empirical data and use mathematical tools for data analysis should be part of the learning at all levels.  

**Attitudes**  
Attitudes refer to the affective aspects of mathematics learning such as:  
○ Beliefs about mathematics and its usefulness  
○ Interest and enjoyment in learning mathematics  
○ Appreciation of the beauty and power of mathematics  
○ Confidence in using mathematics  
○ Perseverance in solving a problem  
Students’ attitudes towards mathematics are shaped by their learning experiences. Making the learning of mathematics fun, meaningful and relevant goes a long way to inculcating positive attitudes towards the subject. Care and attention should be given to the design of the learning activities, to build confidence in and develop appreciation for the subject.  

**Metacognition**  
Metacognition, or “thinking about thinking”, refers to the awareness of, and the ability to control one’s thinking processes, in particular the selection and use of problem-solving strategies. It includes monitoring of one’s own thinking, and self-regulation of learning.  

The provision of metacognitive experience is necessary to help students develop their problem solving abilities. The following activities may be used to develop the metacognitive awareness of students and to enrich their metacognitive experience:  
○ Expose students to general problem solving skills, thinking skills and heuristics, and how these skills can be applied to solve problems.  
○ Encourage students to think aloud the strategies and methods they use to solve particular problems.
- Provide students with problems that require planning (before solving) and evaluation (after solving).
- Encourage students to seek alternative ways of solving the same problem and to check the appropriateness and reasonableness of answer.
- Allow students to discuss how to solve a particular problem and to explain the different methods that they use for solving the problem.

**PEDAGOGY**

The following is a suggested pedagogical framework for teachers that can serve as a coherent pedagogical spine guiding teaching and learning.

**Three Phases of Learning**

Instruction can be thought of as having three common phases: introduction, main activity and conclusion or application (Marsh, 2004). These instruction phases correspond to the 3 learning phases — readiness to learn, engagement in learning, and mastery of learning. These phases together provide a pedagogical framework within which teachers can select and use pedagogies such as cooperative learning, catering to learning styles, consideration for multiple intelligences, etc.

**Introduction Phase — Readiness to Learn**

Getting students ready to learn involves addressing both the affective as well as the cognitive states.

Teachers must be aware of the mathematics experience of the students. Students who have had positive learning experience in mathematics tend to be more ready to learn. On the other hand, students who have had negative learning experiences in mathematics may develop poor self-concept in mathematics that would hamper their readiness to learn. Language difficulties and problems outside schools could also affect the readiness to learn. The teacher’s role is to provide a safe and motivating environment for students to learn.

Teachers must be aware of the pre-requisite knowledge and skills of the students. Without knowing what the students already know, it would be difficult to build and connect cognitively. Not all students recall and understand everything that had been previously taught, therefore surfacing prior knowledge and skills is an important first step in getting students to be ready to learn new knowledge and skills.

**Main Activity Phase — Engagement in Learning**

This is the main phase of learning where teachers use a repertoire of pedagogies and strategies to engage students in learning new knowledge and skills.

Students have different learning styles and they learn best through different approaches. A skilful teacher would structure the activities, taking into account, these differences. Knowledge of multiple intelligences would also be useful to teachers when they design their lessons. In general, students learn best when they are actively engaged in doing mathematics, with opportunities for communication, collaboration, creative work, and
getting appropriate and regular feedback. The concrete to pictorial to abstract sequence of concept development is also relevant here. Concrete experiences can take the form of activities, real-life contexts, or use of manipulatives.

Questions are an important part of the teacher-student interactions in class. Good questioning techniques promote inquiry and higher order thinking. Questions provide opportunities for students to express opinions, explain, clarify, elaborate, and build on each other’s ideas. Open-ended questions can also stimulate thinking, discussions, and reflections. The level of student engagement often depends on the type of questions being asked. Research has shown that appropriate wait-time after asking questions could lead to higher order response to questions.

A well organized learning environment also contributes greatly to students’ engagement in learning. Such an environment includes having ample instructional materials, and well-rehearsed class routines for managing class discipline, group-work, and instructional resources.

**Conclusion or Application Phase – Mastery of Learning**

Practice makes perfect.

Students must be given opportunities to practice the new knowledge and skills in order to achieve mastery. The key to mastery in mathematics learning is often through motivated practice. Improvement occurs incrementally through revisiting the content in smaller as well as novel context and in ways that sustain motivation.

Traditional pen-and-paper exercises are the simplest forms of practice, but they may not be the most motivating and should not be over-used. They should be balanced with other ways of practicing so as to maintain students’ motivation and help them gain new insights. Some of these practices could be in the form of games that are built around new knowledge and skills and students practice these new knowledge and skills in fun and challenging ways.

The range of practice activities can also include new applications or challenges, a journaling activity to reflect and make new connections, or a class presentation for sharing and reinforcing understanding. If students can also see that what they learn in the mathematics classroom is relevant, meaningful and useful to their daily life, then motivated practice can happen beyond the classroom.

Memory techniques such as mind-maps and mnemonics can be used to help students remember the content. Teachers can also make learning more meaningful by helping students to mind-map the connections between mathematics ideas, and between mathematics and other subjects, and also how mathematics is used in the real world.

**Reference**