Categorical models of homotopy type theory

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Recall:

<table>
<thead>
<tr>
<th>homotopy type theory</th>
<th>$\leftrightarrow$</th>
<th>$(\infty, 1)$-categories</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\times$, $+$ types</td>
<td>$\leftrightarrow$</td>
<td>products, coproducts</td>
</tr>
<tr>
<td>equality types ($x = y$)</td>
<td>$\leftrightarrow$</td>
<td>diagonals</td>
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<tr>
<td>$\prod$ types</td>
<td>$\leftrightarrow$</td>
<td>local cartesian closure</td>
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<tr>
<td>univalent universe Type</td>
<td>$\leftrightarrow$</td>
<td>object classifier</td>
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</table>
Two kinds of equality

Problem

Type theory is stricter than $(\infty, 1)$-categories.

In type theory, we have two kinds of “equality”:

1. Equality witnessed by inhabitants of equality types (= paths).
2. Computational equality: $(\lambda x.b)(a)$ evaluates to $b[a/x]$.

These play different roles: type checking depends on computational equality.

- if $a$ evaluates to $b$, and $c : C(a)$, then also $c : C(b)$.
  - In particular, if $a$ evaluates to $b$, then $\text{refl}_b : (a = b)$.
- if $p : (a = b)$ and $c : C(a)$, then only $\text{transport}(p, c) : C(b)$. 
Two kinds of equality

But computational equality is also **strict**er.

**Example**

Composition is computationally strictly associative.

\[
g \circ f \coloneqq \lambda x. g(f(x))
\]

\[
h \circ (g \circ f) = \lambda x. h\left((\lambda x. g(f(x)))\,(x)\right) \rightsquigarrow \lambda x. h(g(f(x)))
\]

\[
(h \circ g) \circ f = \lambda x. \left(\lambda y. h(g(y))\right)(f(x)) \rightsquigarrow \lambda x. h(g(f(x)))
\]

- This is the sort of issue that homotopy theorists are intimately familiar with!
- We need a model for \((\infty, 1)\)-categories with (at least) a strictly associative composition law.
Forget everything you know about homotopy theory; let’s see how the type theorists come at it.

<table>
<thead>
<tr>
<th>Definition</th>
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<tr>
<td>A <strong>display map category</strong> is a category with</td>
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<tr>
<td>• A terminal object.</td>
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<tr>
<td>• A subclass of its morphisms called the <strong>display maps</strong>, denoted $P \to A$ or $P \rightarrow A$.</td>
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<tr>
<td>• Any pullback of a display map exists and is a display map.</td>
</tr>
<tr>
<td>• A display map $P \to A$ is a type dependent on $A$.</td>
</tr>
<tr>
<td>• A display map $A \to 1$ is a plain type (dependent on nothing).</td>
</tr>
<tr>
<td>• Pullback is substitution.</td>
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</tbody>
</table>
Dependent sums of display maps

\[(x : A) \vdash (B(x) : \text{Type})\]

If the types \(B(x)\) are the fibers of \(B \to A\), their dependent sum \(\sum_{x : A} B(x)\) should be the object \(B\).

\[\vdash \left( \sum_{x : A} B(x) : \text{Type} \right)\]
Dependent sums in context

More generally:

\[(x: A), (y : B(x)) \vdash (C(x, y) : Type)\]

\[(x: A) \vdash \left( \sum_{y : B(x)} C(x, y) : Type \right)\]

Dependent sums ←→ display maps compose
• In a category $\mathcal{C}$, if pullbacks along $f : A \to B$ exist, then the functor

$$f^* : \mathcal{C}/B \to \mathcal{C}/A$$

has a left adjoint $\Sigma_f$ given by composition with $f$.

• If $f$ is a display map and display maps compose, then $\Sigma_f$ restricts to a functor

$$(\mathcal{C}/A)_{\text{disp}} \to (\mathcal{C}/B)_{\text{disp}}$$

implementing dependent sums.

• A right adjoint to $f^*$, if one exists, is an “object of sections”. $\mathcal{C}$ is locally cartesian closed iff all such right adjoints $\Pi_f$ exist.
Dependent products of display maps

\[(x : A), (y : B(x)) \vdash (C(x, y) : \text{Type})\]

\[(x : A) \vdash (\prod_{y : B(x)} C(x, y) : \text{Type})\]

Dependent products $\leftrightarrow$ “display maps exponentiate”
The dependent identity type

$$\vdash ((x = y) : Type)$$

must be a display map

$$\text{Id}_A \downarrow \downarrow A \times A$$
The reflexivity constructor

\[(x : A) \vdash (\text{refl}(x) : (x = x))\]

must be a section

\[
\begin{tikzcd}
\Delta^* \text{Id}_A \ar[r] \ar[d] 
& \text{Id}_A \ar[d] \\
A \ar[r, shift right={.5ex}] \ar[ur, shift left={.5ex}] 
& A \times A
\end{tikzcd}
\]

or equivalently a lifting

\[
\begin{tikzcd}
\text{Id}_A \\
A \ar[r, shift right={.5ex}] \ar[ur, shift left={.5ex}] 
& A \times A
\end{tikzcd}
\]
The eliminator says given a dependent type with a section

\[
\begin{array}{c}
\text{refl}^* C \longrightarrow C \\
\downarrow \downarrow \downarrow \\
A \xrightarrow{\text{refl}} \text{Id}_A \\
\end{array}
\]

there exists a compatible section

\[
\begin{array}{c}
C \\
\downarrow \downarrow \\
\text{Id}_A \\
\end{array}
\]

In other words, we have the lifting property

\[
\begin{array}{c}
A \longrightarrow C \\
\downarrow \text{refl} \downarrow \downarrow \\
\text{Id}_A \equiv \text{Id}_A \\
\end{array}
\]
In fact, refl has the left lifting property w.r.t. all display maps.

\[
\begin{align*}
A & \xrightarrow{\text{refl}} f^*C \\
\downarrow & \downarrow \\
\text{Id}_A & \xrightarrow{\exists} C
\end{align*}
\]

\[
\begin{align*}
\text{Id}_A & \xrightarrow{f} B \\
\downarrow & \downarrow \\
\text{Id}_A & \xrightarrow{\text{refl}} A
\end{align*}
\]

Conclusion

Identity types factor $\Delta : A \to A \times A$ as

\[
A \xrightarrow{\text{refl}} \text{Id}_A \xrightarrow{q} A \times A
\]

where $q$ is a display map and refl lifts against all display maps.
Weak factorization systems

Definition

We say $j \ll f$ if any commutative square

\[
\begin{array}{ccc}
X & \longrightarrow & B \\
\downarrow j & & \downarrow f \\
Y & \longrightarrow & A
\end{array}
\]

admits a (non-unique) diagonal filler.

- $\mathcal{J} \ll = \{ f \mid j \ll f \text{ } \forall j \in \mathcal{J} \}$
- $\ll \mathcal{F} = \{ j \mid j \ll f \text{ } \forall f \in \mathcal{F} \}$

Definition

A weak factorization system in a category is $(\mathcal{J}, \mathcal{F})$ such that

1. $\mathcal{J} = \ll \mathcal{F}$ and $\mathcal{F} = \mathcal{J} \ll$.
2. Every morphism factors as $f \circ j$ for some $f \in \mathcal{F}$ and $j \in \mathcal{J}$. 
Theorem (Gambino–Garner)

In a display map category that models identity types, any morphism \( g : A \rightarrow B \) factors as

\[
A \xrightarrow{j} Ng \xrightarrow{f} B
\]

where \( f \) is a display map, and \( j \) lifts against all display maps.

\[
(y : B) \vdash Ng(y) := \text{hfiber}(g, y) := \sum_{x : A} (g(x) = y)
\]

is the type-theoretic mapping path space.
The identity type wfs

Corollary (Gambino-Garner)

In a type theory with identity types,

$$\left( \blacksquare (\text{display maps}), (\blacksquare (\text{display maps}))\blacksquare \right)$$

is a weak factorization system.

This behaves very much like (acyclic cofibrations, fibrations):

- Dependent types are like fibrations (recall “transport”).
- Every map in $\blacksquare (\text{display maps})$ is an equivalence; in fact, the inclusion of a deformation retract.
Conversely:

**Theorem (Awodey–Warren, Garner–van den Berg)**

In a display map category, if

\[
\left(\Box (\text{display maps}), (\Box (\text{display maps}))\Box \right)
\]

is a “pullback-stable” weak factorization system, then the category \((\text{almost}^* )\) models identity types.

\[
\text{identity types} \quad \leftrightarrow \quad \text{weak factorization systems}
\]
### Definition (Quillen)

A **model category** is a category $\mathcal{C}$ with limits and colimits and three classes of maps:

- $\mathcal{C} = $ cofibrations
- $\mathcal{F} = $ fibrations
- $\mathcal{W} = $ weak equivalences

such that

1. $\mathcal{W}$ has the 2-out-of-3 property.
2. $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are weak factorization systems.
Corollary

Let $\mathcal{M}$ be a model category such that

1. $\mathcal{M}$ (as a category) is locally cartesian closed.
2. $\mathcal{M}$ is right proper.
3. The cofibrations are the monomorphisms.

Then $\mathcal{M}$ (almost\(^*\)) models type theory with dependent sums, dependent products, and identity types.

Examples

- Simplicial sets with the Quillen model structure.
- Any injective model structure on simplicial presheaves.
Homotopy type theory in categories

\[(x : A) \vdash p : \text{isProp}(B(x))\]

\[\iff (x : A), (u : B(x)), (v : B(x)) \vdash (p_{u,v} : (u = v))\]

\[\iff \text{The path object } P_{A}B \text{ has a section in } \mathcal{M}/A\]

\[\iff \text{Any two maps into } B \text{ are homotopic over } A\]

\[(x : A) \vdash p : \text{isContr}(B(x))\]

\[\iff (x : A) \vdash p : \text{isProp}(B(x)) \times B(x)\]

\[\iff \text{Any two maps into } B \text{ are homotopic over } A\]

\[\iff \text{and } B \to A \text{ has a section}\]

\[\iff B \to A \text{ is an acyclic fibration}\]
Homotopy type theory in categories

For $f: A \to B$,

$$\vdash p: \text{isEquiv}(f) \iff \vdash \prod_{y : B} \text{isContr}(\text{hfiber}(f, y))$$

$$\iff (y : B) \vdash \text{isContr}(\text{hfiber}(f, y))$$

$$\iff \text{hfiber}(f) \to A \text{ is an acyclic fibration}$$

$$\iff f \text{ is a (weak) equivalence}$$

(Recall $\text{hfiber}$ is the factorization $A \to Nf \to B$ of $f$.)

Conclusion

Any theorem about “equivalences” that we can prove in type theory yields a conclusion about weak equivalences in appropriate model categories.
Coherence

Another Problem

Type theory is even stricter than 1-categories!

Recall that substitution is pullback.

\[
\begin{array}{ccc}
  f^* g^* A & \longrightarrow & g^* P \\
  \downarrow & & \downarrow \\
  A & \longrightarrow & B
\end{array}
\]

\[
\begin{array}{ccc}
  & & P \\
  & & \downarrow \\
  & & C
\end{array}
\]

\[
f: A \to B \quad g: B \to C
\]

\[
a: A \vdash P(g(f(a))) \\
b: B \vdash P(g(b)) \\
c: C \vdash P(c)
\]
Another Problem

Type theory is even stricter than 1-categories!

Recall that substitution is pullback.

$$(g \circ f)^* A \to P$$

\[ a: A \vdash P(g(f(a))) \]

But, of course, $f^* g^* P$ is only isomorphic to $(g \circ f)^* P$. 
Coherence with a universe

There are several resolutions; perhaps the cleanest is:

**Solution (Voevodsky)**

Represent dependent types by their *classifying maps* into a universe object.

Now substitution is *composition*, which is strictly associative (in our model category):

\[
A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{P} U
\]

\[
A \xrightarrow{g \circ f} C \xrightarrow{P} U
\]

We needed a universe object anyway, to model the type Type and prove univalence.

**New problem**

Need *very strict models* for universe objects.
Representing fibrations

(Following Kapulkin–Lumsdaine–Voevodsky)

Goal

A universe object in simplicial sets giving coherence and univalence.

Simplicial sets are a presheaf category, so there is a standard trick to build representing objects.

\[ U_n \cong \text{Hom}(\Delta^n, U) \cong \{\text{fibrations over } \Delta^n\} \]

But \( n \mapsto \{\text{fibrations over } \Delta^n\} \) is only a pseudofunctor; we need to rigidify it.
A technical device (Voevodsky)

A well-ordered Kan fibration is a Kan fibration $p: E \to B$ together with, for every $x \in B_n$, a well-ordering on $p^{-1}(x) \subseteq E_n$.

Two well-ordered Kan fibrations are isomorphic in at most one way which preserves the orders.

Definition

\[
U_n := \left\{ X \to \Delta^n \text{ a well-ordered fibration} \right\} \mathrel{\overset{\text{ordered}}{\sim}}
\]
\[
\tilde{U}_n := \left\{ (X, x) \mid X \to \Delta^n \text{ well-ordered fibration, } x \in X_n \right\} \mathrel{\overset{\text{ordered}}{\sim}}
\]

(with some size restriction, to make them sets).
The universal Kan fibration

**Theorem**

The forgetful map $\tilde{U} \to U$ is a Kan fibration.

**Proof.**

A map $E \to B$ is a Kan fibration if and only if every pullback

$$
\begin{array}{ccc}
\Delta^n & \xrightarrow{b} & B \\
\downarrow & & \downarrow \\
\Lambda^n_k & \hookrightarrow & \Delta^n \\
\end{array}
$$

is such, since the horns $\Lambda^n_k \hookrightarrow \Delta^n$ have codomain $\Delta^n$.

Thus, of course, every pullback of $\tilde{U} \to U$ is a Kan fibration.
The universal Kan fibration

Theorem

*Every (small) Kan fibration $E \to B$ is some pullback of $\tilde{U} \to U$:* 

\[
\begin{array}{ccc}
E & \to & \tilde{U} \\
\downarrow & & \downarrow \\
B & \to & U
\end{array}
\]

Proof.

Choose a well-ordering on each fiber, and map $x \in B_n$ to the isomorphism class of the well-ordered fibration $b^*(E) \to \Delta^n$.

It is essential that we have *actual* pullbacks here, not just homotopy pullbacks.
Let the size-bound for $U$ be inaccessible (a Grothendieck universe). Then small fibrations are closed under all categorical constructions.

Now we can interpret type theory with coherence, using morphisms into $U$ for dependent types.

**Example**

A context

$$(x : A), (y : B(x)), (z : C(x, y))$$

becomes a sequence of fibrations together with classifying maps:

$$\xymatrix{ C \ar[r] & B \ar[r] & A \ar[r] & 1 \\
\sim U \ar[r] & U & \sim U \ar[r] & U & \sim U \ar[r] & U \ar[r] & 1 }
$$

in which each trapezoid is a pullback.
**Strict cartesian products**

Every type-theoretic operation can be done once over $U$, then implemented by composition.

**Example (Cartesian product)**

- Pull $\tilde{U}$ back to $U \times U$ along the two projections $\pi_1, \pi_2$.
- Their fiber product over $U \times U$ admits a classifying map:

\[
\begin{array}{c}
\text{(} \pi_1^* \tilde{U} \text{)} \times U \times U \text{ (} \pi_2^* \tilde{U} \text{)} \rightarrow \tilde{U} \\
\downarrow \hspace{2cm} \downarrow \\
U \times U \rightarrow \tilde{U} \\
\end{array}
\]

- Define the product of $[A]: X \rightarrow U$ and $[B]: X \rightarrow U$ to be

\[
X \xrightarrow{([A],[B])} U \times U \xrightarrow{[\times]} U
\]

This has strict substitution.
Problem

So far the object $U$ lives outside the type theory. We want it inside, giving a universe type “Type” and univalence.

Solution

Let $U'$ be a bigger universe. If $U$ is $U'$-small and fibrant, then it has a classifying map:

\[
\begin{array}{ccc}
U & \longrightarrow & \tilde{U}' \\
\downarrow & & \downarrow \\
1 & \overset{u}{\longrightarrow} & U'
\end{array}
\]

and the type theory defined using $U'$ has a universe type $u$. 

\[\]
Theorem

$U$ is fibrant.

Outline of proof.

\[
\begin{array}{ccc}
\Lambda^n_k & \xrightarrow{f} & U \\
\downarrow & & \downarrow \\
\Delta^n & \xleftarrow{j} & ? \\
\end{array}
\]

With hard work, we can extend $f^*\tilde{U}$ to a fibration over $\Delta^n$:

\[
\begin{array}{ccc}
f^*\tilde{U} & \rightarrow & P \\
\downarrow & & \downarrow \\
\Lambda^n_k & \xrightarrow{j} & \Delta^n \\
\end{array}
\]

and extend the well-ordering of $f^*\tilde{U}$ to $P$, yielding $g: \Delta^n \rightarrow U$ with $gj = f$ (and $g^*\tilde{U} \cong P$).
Extending fibrations

**Lemma**

Any fibration $P \to \Lambda^n_k$ is the pullback of some fibration over $\Delta^n$.

**Proof.**

- Let $P' \subseteq P$ be a minimal subfibration.
- There is a retraction $P \to P'$ that is an acyclic fibration.
- Since $\Lambda^n_k$ is contractible, the minimal fibration $P' \to \Lambda^n_k$ is isomorphic to a trivial bundle $\Lambda^n_k \times F \to \Lambda^n_k$.

\[
\begin{array}{ccc}
P & \longrightarrow & \Pi_{j \times F} P \\
\downarrow & & \downarrow \\
P' \cong \Lambda^n_k \times F & \longrightarrow & \Delta^n \times F \\
\downarrow & & \downarrow \\
\Lambda^n_k & \longrightarrow & \Delta^n \\
\end{array}
\]
We want to show that $PU \to \text{Eq}(U)$ is an equivalence:

It suffices to show:

1. The composite $U \to \text{Eq}(U)$ is an equivalence.
2. The projection $\text{Eq}(U) \to U$ is an equivalence.
3. The projection $\text{Eq}(U) \to U$ is an acyclic fibration.
Univalence

By representability, a commutative square with a lift

\[
\begin{array}{ccc}
\partial \Delta^n & \rightarrow & \text{Eq}(U) \\
i & & \downarrow \\
\Delta^n & \rightarrow & U
\end{array}
\]

corresponds to a diagram

\[
\begin{array}{ccc}
E_1 & \rightarrow & \overline{E}_1 \\
& \downarrow & \downarrow \\
& E_2 & \rightarrow \overline{E}_2 \\
\partial \Delta^n & \rightarrow & \Delta^n \\
i & \downarrow & \downarrow
\end{array}
\]

with \( E_1 \rightarrow E_2 \) an equivalence and \( \overline{E}_1 \rightarrow \overline{E}_2 \) equivalences.
• By factorization, consider separately the cases when $E_1 \to E_2$ is (1) an acyclic fibration or (2) an acyclic cofibration.
• (1) $\overline{E}_1 \to \overline{E}_2$ is an acyclic fibration ($\Pi_i$ preserves such).
• (2) $\overline{E}_1$ is a deformation retract of $\overline{E}_2$. 

Univalence
Definition

An \((\infty, 1)\)-topos is an \((\infty, 1)\)-category that is a left-exact localization of an \((\infty, 1)\)-presheaf category.

Examples

- \(\infty\)-groupoids (plays the role of the 1-topos \(\text{Set}\))
- Parametrized homotopy theory over any space \(X\)
- \(G\)-equivariant homotopy theory for any group \(G\)
- \(\infty\)-sheaves/stacks on any space
- “Smooth \(\infty\)-groupoids” (or “algebraic” etc.)
Definition (Rezk)

An **object classifier** in an \((\infty, 1)\)-category \(C\) is a morphism \(\tilde{U} \to U\) such that pullback

\[
\begin{array}{ccc}
B & \longrightarrow & \tilde{U} \\
\downarrow & & \downarrow \\
A & \longrightarrow & U
\end{array}
\]

induces an equivalence of \(\infty\)-groupoids

\[
\text{Hom}(A, U) \simeq \text{Core}(C/A)_{\text{small}}
\]

(“Core” is the maximal sub-\(\infty\)-groupoid.)
Theorem (Rezk)

An $(\infty, 1)$-category $\mathcal{C}$ is an $(\infty, 1)$-topos if and only if

1. $\mathcal{C}$ is locally presentable.
2. $\mathcal{C}$ is locally cartesian closed.
3. $\kappa$-compact objects have object classifiers for $\kappa \gg 0$.

Corollary

If a combinatorial model category $\mathcal{M}$ interprets dependent type theory as before (i.e. it is locally cartesian closed, right proper, and the cofibrations are the monomorphisms), and contains universes for $\kappa$-compact objects that satisfy the univalence axiom, then the $(\infty, 1)$-category that it presents is an $(\infty, 1)$-topos.
Conjecture

Every $(\infty, 1)$-topos can be presented by a model category which interprets dependent type theory with the univalence axiom.

Homotopy type theory is the internal logic of $(\infty, 1)$-toposes.

If this is true, then anything we prove in homotopy type theory (which we can also verify with a computer) will automatically be true internally to any $(\infty, 1)$-topos. The “constructive core” of homotopy theory should be provable in this way, in a uniform way for “all homotopy theories”. 
Status of the conjecture

\[ \infty Gpd \longrightarrow (\infty, 1)\text{-presheaves} \longrightarrow (\infty, 1)\text{-toposes} \]

\[ \begin{array}{ccc}
\infty Gpd & \longrightarrow & (\infty, 1)\text{-presheaves} \\
& \downarrow & \uparrow \\
\text{inverse } (\infty, 1)\text{-presheaves} & & (\infty, 1)\text{-toposes}
\end{array} \]