Additive ∞-categories and canonical monoidal structures I

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REFERENCE

joint with *David Gepner* and *Thomas Nikolaus* (arXiv):

‘*Universality of multiplicative infinite loop space machines*’

Plan

Calibration of notation and terminology

Notation and terminology as in *Higher Topos Theory* and *Higher Algebra*.

∞-category = simplicial set with inner-horn-filling-property

Alternative choices:

- quasi-category
- weak Kan complex
- inner Kan complex
- quategory
- ...

$\mathbb{E}_\infty$-MONOIDS IN ∞-CATEGORIES

Segal's picture on $\mathbb{E}_\infty$-monoids:

$\mathcal{F}\mathrm{in}_*$	category of finite pointed sets, pointed maps
$\langle n\rangle \in \mathcal{F}\mathrm{in}_*$	finite pointed set ({0,1,...,n},0)
$\rho_i\colon \langle n\rangle \to \langle 1\rangle$	pointed map with $\rho_i^{-1}(1) = \{i\}$
$\mathcal{C}$	∞-category with finite products

DEFINITION

An $\mathbb{E}_\infty$-**monoid** in $\mathcal{C}$ is a functor $M\colon N(\mathcal{F}\mathrm{in}_*) \to \mathcal{C}$ such that the canonical maps

$$\rho_*\colon M_n \to M_1 \times \ldots \times M_1, \quad n \geq 0,$$

are equivalences.

$\mathbb{E}_\infty$-groups in ∞-categories

An $\mathbb{E}_\infty$-monoid $M\colon N(\mathcal{F}\mathrm{in}_*) \to \mathcal{C}$ gives rise to

- multiplication maps $\mu\colon M_1 \times M_1 \to M_1$,
- **shear maps** $\sigma = (\mu, \pi_2)\colon M_1 \times M_1 \to M_1 \times M_1$.

Definition

An $\mathbb{E}_\infty$-monoid in $\mathcal{C}$ is an $\mathbb{E}_\infty$**-group** if equivalently:

1. There is an **inversion** map $M_1 \to M_1$.
2. The shear maps are equivalences.
3. The underlying commutative monoid in $\mathrm{Ho}(\mathcal{C})$ is a group object.

Preadditive ∞-categories

Definition

Let $\mathcal{C}$ be a pointed ∞-category with finite coproducts and finite products. Then $\mathcal{C}$ is **preadditive** if the canonical maps $X \sqcup Y \to X \times Y$ are equivalences.

For such pointed ∞-category $\mathcal{C}$ with finite coproducts and finite products the following are equivalent:

1. The ∞-category $\mathcal{C}$ is preadditive.
2. The homotopy category $\mathrm{Ho}(\mathcal{C})$ is preadditive.
3. The forgetful functor $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C}) \to \mathcal{C}$ is an equivalence.

Preadditive ∞-categories II

Example

If $\mathcal{C}$ has finite products, then $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C})$ is preadditive.

Corollary

If $\mathcal{C}$ has finite products, then the forgetful functor $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C})) \to \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C})$ is an equivalence.

In preadditive $\mathcal{C}$, canonical $\mathbb{E}_\infty$-monoid structures arise from fold maps $\nabla\colon X \oplus X \to X$. Associated to these we have shear maps $\sigma = (\nabla, \pi_2)\colon X \oplus X \to X \oplus X$.

ADDITIVE ∞-CATEGORIES

DEFINITION

A preadditive ∞-category is **additive** if the shear maps $\sigma: X \oplus X \to X \oplus X$ are equivalences.

PROPOSITION

1. *A preadditive ∞-category $\mathcal{C}$ is additive iff $\mathrm{Ho}(\mathcal{C})$ is additive iff $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C}) \to \mathcal{C}$ is an equivalence.*
2. *If $\mathcal{C}$ has finite products, then $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C})$ is additive and the functor $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C})) \to \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C})$ is an equivalence.*

Presentability of monoids and groups

Let $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ be the ∞-category of presentable ∞-categories with morphisms the left adjoint functors.

Proposition

Let $\mathcal{C}$ be a presentable ∞-category. Then the ∞-categories $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C})$ and $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C})$ are presentable.

- $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C})$ is an accessible localization of $\mathcal{C}^{N(\mathcal{F}\mathrm{in}_*)}$.
- $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C})$ is an accessible localization of $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C})$.

Corollary (Group completion)

Given $\mathcal{C} \in \mathcal{P}\mathrm{r}^{\mathrm{L}}$ then there are adjunctions:

$$\mathcal{C} \rightleftarrows \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C}) \rightleftarrows \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C})$$

$\mathrm{Mon}_{\mathbb{E}_\infty}(-)$, $\mathrm{Grp}_{\mathbb{E}_\infty}(-)$ AS LOCALIZATIONS

The assignments $\mathcal{C} \mapsto \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C})$ and $\mathcal{C} \mapsto \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C})$ are obviously functorial in product-preserving functors.

Proposition

The assignments $\mathcal{C} \mapsto \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C})$ and $\mathcal{C} \mapsto \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C})$ define functors $\mathrm{Mon}_{\mathbb{E}_\infty}(-)\colon \mathcal{P}r^{\mathrm{L}} \to \mathcal{P}r^{\mathrm{L}}$ and $\mathrm{Grp}_{\mathbb{E}_\infty}(-)\colon \mathcal{P}r^{\mathrm{L}} \to \mathcal{P}r^{\mathrm{L}}$.

Theorem

1. *The functor $\mathrm{Mon}_{\mathbb{E}_\infty}(-)\colon \mathcal{P}r^{\mathrm{L}} \to \mathcal{P}r^{\mathrm{L}}$ is a localization with local objects the preadditive, presentable ∞-categories.*
2. *The functor $\mathrm{Grp}_{\mathbb{E}_\infty}(-)\colon \mathcal{P}r^{\mathrm{L}} \to \mathcal{P}r^{\mathrm{L}}$ is a localization with local objects the additive, presentable ∞-categories.*

SOME IMMEDIATE CONSEQUENCES

Let $\mathcal{C}, \mathcal{D}$ be presentable ∞-categories, and let $\mathcal{S}$ be the ∞-category of spaces ('free homotopy theory on Δ^0').

COROLLARY ('PREADDITIVIZATION')

If $\mathcal{D}$ is preadditive, then $\mathcal{C} \to \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C})$ induces a canonical equivalence $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$. In particular, we obtain $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S}), \mathcal{D}) \simeq \mathcal{D}$.

COROLLARY ('ADDITIVIZATION')

If $\mathcal{D}$ is additive, then $\mathcal{C} \to \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C})$ induces a canonical equivalence $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$. In particular, we obtain $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S}), \mathcal{D}) \simeq \mathcal{D}$.

A refined picture of the stabilization

$\mathcal{P}r^{L}_{Pt}$	*pointed* presentable ∞-categories
$\mathcal{P}r^{L}_{Pre}$	*preadditive* presentable ∞-categories
$\mathcal{P}r^{L}_{Add}$	*additive* presentable ∞-categories
$\mathcal{P}r^{L}_{St}$	*stable* presentable ∞-categories

Theorem (Stabilization)

1. *The stabilization of presentable ∞-categories factors as* $\mathcal{P}r^{L} \rightleftarrows \mathcal{P}r^{L}_{Pt} \rightleftarrows \mathcal{P}r^{L}_{Pre} \rightleftarrows \mathcal{P}r^{L}_{Add} \rightleftarrows \mathcal{P}r^{L}_{St}$.
2. *In particular, for $\mathcal{C} \in \mathcal{P}r^{L}$, the suspension spectrum functor factors as*

$$\Sigma^{\infty}_{+} : \mathcal{C} \to \mathcal{C}_{*} \to \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C}) \to \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C}).$$

∞-CATEGORICAL 'DIRECT SUM' K-THEORY

Classical picture:

$\mathcal{C}$	symmetric, monoidal category
$\widetilde{\mathcal{C}}$	largest subgroupoid of $\mathcal{C}$
$\lvert\widetilde{\mathcal{C}}\rvert$	associated $\mathbb{E}_\infty$-space
$K(\mathcal{C})$	K-theory spectrum via group-completion

Some steps towards an ∞-categorical variant are:

1. The inclusion $\mathcal{S} = \mathcal{G}\mathrm{rpd}_\infty \to \mathcal{C}\mathrm{at}_\infty$ admits a right adjoint given by $\mathcal{C} \mapsto \widetilde{\mathcal{C}}$.
2. The ∞-categories $\mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at}_\infty$ and $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C}\mathrm{at}_\infty)$ are equivalent, compatibly with ∞-groupoids.

DEFINITION

The **algebraic K-theory** $\mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at}_\infty \to \mathrm{Sp}$ is defined as the composition

$$\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C}\mathrm{at}_\infty) \to \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S}) \to \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S}) \to \mathrm{Sp}(\mathcal{S}).$$

Will be discussed further by *Thomas Nikolaus* in the second part of this talk!

Thanks for your attention!