Categorical Aspects of Type Theory

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Introduction

Motivation: To understand Martin-Löf type theory.

Conceptual mathematics $\rightarrow$ category theory.

Two questions:

- Is type theory soluble in category theory?
- Is category theory soluble in type theory?

I will not discuss the second question here.

Overview

Aspects of categorical logic:

- Locally cartesian closed categories
- Tribes

Homotopical logic:

- Weak factorization systems
- Homotopical algebra
- Pre-typoi
- Typoi
- Univalent typoi

Aspects of categorical logic

The basic principles of categorical logic was expressed in Lawvere's paper **Adjointness in Foundation** (1969). I will use these principles implicitly.

- Locally cartesian closed categories
- Tribes
- Π-tribes

Terminal objects and terms

Recall that an object $\top$ in a category $\mathcal{C}$ is said to be **terminal** if for every object $A \in \mathcal{C}$, there is a unique map $A \to \top$.

If $\top$ is a terminal object, then a map $u : \top \to A$ is called

- a **global section** of the *object* A, $\quad u \in \Gamma(A)$
- an **element** of A, $\quad u \in A$
- a **constant** of *sort* A, $\quad u \in A$
- a **term** of **type** A, $\quad u : A$.

Cartesian product

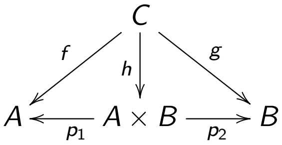
Recall that the **cartesian product** $A \times B$ of two objects A and B in a category $\mathcal{C}$ is an object $A \times B$ equipped with a pair of *projection*

$$A \xleftarrow{p_1} A \times B \xrightarrow{p_2} B$$

having the following universal property: for any object $C \in \mathcal{C}$ and any a pair of maps

$$A \xleftarrow{f} C \xrightarrow{g} B,$$

there is a unique map $h = \langle f, g \rangle : C \to A \times B$ such that $p_1 h = f$ and $p_2 h = g$.



Cartesian category

The map $h \mapsto (p_1 h, p_2 h)$ is a natural bijection between

$$\frac{\text{the maps } C \to A \times B}{\text{and the pairs of maps } C \to A, \; C \to B.}$$

A category $\mathcal{C}$ is **cartesian** if it has (binary) cartesian product and a terminal object $\top$.

Equivalently, a category $\mathcal{C}$ is cartesian if it has finite cartesian products.

Exponential

Let A and B be two objects of a cartesian category $\mathcal{C}$.

An object $[A, B]$ equipped with a map $ev : [A, B] \times A \to B$ is called the **exponential** of B by A if:

for every object $C \in \mathcal{C}$ and every map $f : C \times A \to B$

there exists a unique map $\ulcorner f \urcorner : C \to [A, B]$ such that

$$\begin{array}{ccc} C \times A & \xrightarrow{\ulcorner f \urcorner \times A} & [A, B] \times A \\ & \searrow^{f} & \downarrow{\scriptstyle ev} \\ & & B \end{array}$$

We write $\lambda^A f := \ulcorner f \urcorner$.

Cartesian closed categories

The map $f \mapsto \lambda^A f$ is a natural bijection between

$$\frac{\text{the maps } \; C \times A \to B}{\text{and the maps } \; C \to [A, B].}$$

A cartesian category $\mathcal{C}$ is said to be **closed** if the object $[A, B]$ exists for every pair of objects $A, B \in \mathcal{C}$.

A cartesian category $\mathcal{C}$ is closed if and only if the functor

$$A \times (-) : \mathcal{C} \to \mathcal{C}$$

has a right adjoint $[A, -]$ for every object $A \in \mathcal{C}$.

Cartesian closed categories(2)

Examples of cartesian closed categories

- the category of sets **Set**
- the category of (small) catégories **Cat** (Lawvere)
- the category of groupoids **Grpd**
- Every category $[\mathbb{C}, \mathbf{Set}]$
- The category of simplicial sets $[\Delta^{op}, \mathbf{Set}]$

Cartesian closed categories and lambda calculus

Every cartesian category $\mathcal{A}$ generates freely a cartesian closed category $CC[\mathcal{A}]$.

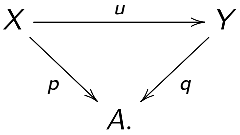
- the morphism in $CC[\mathcal{A}]$ are represented by **lambda terms**;
- lambda terms have a **normal form**;
- the category $CC[\mathcal{A}]$ is **decidable** if $\mathcal{A}$ is decidable.

Lambek and Scott: *Higher categorical logic*.

Slice categories

Recall that the *slice category* $\mathcal{C}/A$ has for objects the pairs (X, p), where p is a map $X \to A$ in $\mathcal{C}$. The map $p : X \to A$ is called the *structure map* of (X, p).

A morphism $(X, p) \to (Y, q)$ in $\mathcal{C}/A$ is a map $u : X \to Y$ in $\mathcal{C}$ such that $qu = p$,



Push-forward

To every map $f : A \to B$ in a category $\mathcal{C}$ we can associate a **push-forward** functor

$$f_! : \mathcal{C}/A \to \mathcal{C}/B$$

by putting $f_!(X, p) = (X, fp)$ for every map $p : X \to A$,

$$\begin{array}{ccc} X & = & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle fp} \\ A & \xrightarrow{f} & B. \end{array}$$

Pull-back

Recall that the **fiber product** of two maps $X \to A$ and $Y \to A$ in a category $\mathcal{C}$ is their cartesian product $X \times_A Y$ as objects of the category $\mathcal{C}/A$.

$$\begin{array}{ccc} X \times_A Y & \xrightarrow{p_2} & Y \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{p} & A. \end{array}$$

The square is also called a **pullback square**.

Base changes

In a category with finite limits $\mathcal{C}$ the push-forward functor $f_! : \mathcal{C}/A \to \mathcal{C}/B$ has a right adjoint

$$f^\star : \mathcal{C}/B \to \mathcal{C}/A$$

for any map $f : A \to B$. The functor $f^\star$ takes a map $p : X \to B$ to the map $p_1 : A \times_B X \to A$ in a pullback square

$$\begin{array}{ccc} A \times_B X & \xrightarrow{p_2} & X \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & B. \end{array}$$

The map p_1 is said to be the **base change** of the map $p : X \to B$ *along* the map $f : A \to B$.

Locally cartesian closed categories(1)

A category with finite limits $\mathcal{C}$ is said to be **locally cartesian closed** (lcc) if the category $\mathcal{C}/A$ is cartesian closed for every object $A \in \mathcal{C}$.

A category with finite limits $\mathcal{C}$ is lcc if and only if the base change functor $f^* : \mathcal{C}/B \to \mathcal{C}/A$ has a right adjoint

$$f_\star : \mathcal{C}/A \to \mathcal{C}/B$$

for every map $f : A \to B$ in $\mathcal{C}$.

Locally cartesian closed categories(2)

If $X = (X, p) \in \mathcal{C}/A$, then $f_\star(X) \in \mathcal{C}/B$ is called the **internal product** of X *along* $f : A \to B$, and denoted

$$\Pi_f(X) := f_\star(X).$$

If $Y = (Y, q) \in \mathcal{C}/B$, there is a natural bijection between

$$\frac{\text{the maps} \qquad Y \to \Pi_f(X) \qquad \text{in } \mathcal{C}/B}{\text{and the maps} \qquad f^\star(Y) \to X \qquad \text{in } \mathcal{C}/A}$$

Locally cartesian closed categories(3)

Examples of lcc categories

- the category **Set**
- every category $[\mathbb{C}, \mathbf{Set}]$
- every Grothendieck topos
- every elementary topos

Non-examples

The category **Cat**

The category **Grpd**

Logical functors

Definition

A functor $F : \mathcal{C} \to \mathcal{D}$ between lcc categories is **logical** if it preserves

- finite limits;
- internal products.

The last condition means that the comparison map

$$F\Pi_f(X) \to \Pi_{F(f)}(FX)$$

is an isomorphism for any pair of maps $X \to A$ and $f : A \to B$ in $\mathcal{C}$.

Logical functors(2)

If $\mathcal{C}$ is a locally cartesian closed category, then

- the Yoneda functor $y : \mathcal{C} \to \hat{\mathcal{C}} = [\mathcal{C}^{op}, \mathbf{Set}]$ is logical;
- the base change functor $f^* : \mathcal{C}/B \to \mathcal{C}/A$ is a logical for every map $f : A \to B$ in $\mathcal{C}$.

Generic terms

Let $i : \mathcal{C} \to \mathcal{C}/A$ be the base change functor. By definition, $i(X) = (A \times X, p_1)$ for every $X \in \mathcal{C}$.

Theorem

The functor $i : \mathcal{C} \to \mathcal{C}/A$ is logical and $\mathcal{C}/A$ is obtained from $\mathcal{C}$ by adding freely a term $x_A : i(A)$.

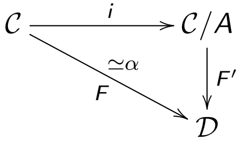
More precisely, $i(\top) = (A, 1_A)$ and $i(A) = (A \times A, p_1)$.

The diagonal $A \to A \times A$ is a map $x_A : i(\top) \to i(A)$.

Generic terms(2)

For any logical functor $F : \mathcal{C} \to \mathcal{E}$ with values in a lcc category $\mathcal{E}$ and any term $a : F(A)$,

there exists a logical functor $F' : \mathcal{C}/A \to \mathcal{D}$ and a natural isomorphism $\alpha : F \simeq F' \circ i$



such that $\alpha_A(a) = F'(x_A)$.

Moreover, the pair (F', α) is unique up to a unique iso of pairs.

Thus, $\mathcal{C}/A = \mathcal{C}[x_A]$ and the term $x_A : i(A)$ is **generic**.

Tribes(0)

A class of maps $\mathcal{F}$ in a category $\mathcal{C}$ is said to be

closed under base changes if

$$X \to B \text{ in } \mathcal{F} \qquad \Longrightarrow \qquad A \times_B X \to A \text{ exists and belong to } \mathcal{F}$$

$$\begin{array}{ccc} A \times_B X & \longrightarrow & X \\ {\scriptstyle \in\mathcal{F}}\downarrow & & \downarrow{\scriptstyle \in\mathcal{F}} \\ A & \xrightarrow{\quad f \quad} & B \end{array}$$

for any map $f : A \to B$ in $\mathcal{C}$

Tribes(1)

Definition

Let $\mathcal{C}$ be a category with terminal object $\top$. We say that a class of maps $\mathcal{F} \subseteq \mathcal{C}$ is a **tribe structure** if the following conditions are satisfied:

- every isomorphism belongs to $\mathcal{F}$;
- $\mathcal{F}$ is closed under composition and base changes;
- the map $X \to \top$ belongs to $\mathcal{F}$ for every object $X \in \mathcal{C}$.

We shall say that the pair $(\mathcal{C}, \mathcal{F})$ is a **tribe**.

A map in $\mathcal{F}$ is a **family** or a **fibration** of the tribe.

Tribes(2)

The **fiber** of a fibration $p : X \to A$ at a point $a : \top \to A$ is the object $X(a)$ defined by the pullback square

$$\begin{array}{ccc} X(a) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \top & \xrightarrow{\;a\;} & A. \end{array}$$

A fibration $p : X \to A$ is an **internal family** $(X(a) : a \in A)$ of objects parametrized by the codomain of p.

Tribes(3)

The full subcategory of $\mathcal{C}/A$ whose objects are the fibrations $X \to A$ is denoted $\mathcal{C}(A)$.

The category $\mathcal{C}(A)$ has the structure of a tribe where a morphism $f : (X, p) \to (Y, q)$ in $\mathcal{C}(A)$ is a *fibration* if $f : X \to Y$ is a fibration in $\mathcal{C}$.

And object of $\mathcal{C}(A)$ is a type which **depends** on the type A.

If $u : A \to B$, then the base change functor $u^\star : \mathcal{C}(B) \to \mathcal{C}(A)$ is an operation of **change of parameters**: we have

$$u^\star(Y)(a) = Y(u(a))$$

for every every fibration $Y \to B$ and every term $a : A$.

Tribes (4)

Definition

A **morphism of tribes** $F : \mathcal{C} \to \mathcal{D}$ is a functor which

- takes fibrations to fibrations;
- preserves base changes of fibrations;
- preserves terminal objects.

For example, the base change functor

$$u^\star : \mathcal{C}(B) \to \mathcal{C}(A)$$

is a morphism of tribes for any map $u : A \to B$ in a tribe $\mathcal{C}$.

Variables=generic terms

The base change functor $i : \mathcal{C} \to \mathcal{C}(A)$ is a morphism of tribes.

Theorem

The tribe $\mathcal{C}(A)$ is obtained from $\mathcal{C}$ by adding freely a term $x_A : i(A)$.

The term $x_A : i(A)$ is **generic**.

Types and contexts

An object $p : E \to A$ of $\mathcal{C}(A)$ is a **type** $E(x)$ in **context** $x : A$.

Type theorists write

$$x : A \vdash E(x) : type$$

where $E(x)$ is the general fiber of the map $p : E \to A$,

$$\begin{array}{ccc} E(x) & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ \top & \xrightarrow{x} & A \end{array}$$

The object E is the **total space** of the fibration $p : E \to A$,

$$E = \sum_{x:A} E(x).$$

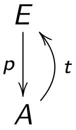
Terms and types

A **term** $t(x)$ of type $E(x)$ is a section t of the map $p : E \to A$.

Type theorists write

$$x : A \vdash t(x) : E(x)$$

Topologists write



Push-forward and sum

To every fibration $f : A \to B$ in tribe $\mathcal{C}$ we can associate a *push-forward* functor

$$f_! : \mathcal{C}(A) \to \mathcal{C}(B)$$

by putting $f_!(E, p) = (E, fp)$,

$$\begin{array}{ccc} E & = & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle fp} \\ A & \xrightarrow{f} & B. \end{array}$$

Formally,

$$f_!(E)(b) = \sum_{f(a)=b} E(a)$$

for every fibration $E \to A$ and every $b : B$.

Push-forward and sum(2)

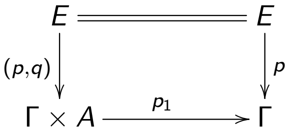
The functor $f_! : \mathcal{C}(A) \to \mathcal{C}(B)$ is left adjoint to the functor $f^\star$.

For very $X \in \mathcal{C}(A)$ and $Y \in \mathcal{C}(B)$, there is a natural bijection between

$$\frac{\text{the maps} \qquad f_!(X) \to Y \qquad \text{in } \mathcal{C}(B)}{\text{and the maps} \qquad X \to f^\star(Y) \qquad \text{in } \mathcal{C}(A)}$$

Sum formation

$$\frac{\Gamma,\ \ x : A \vdash E(x) : type}{\Gamma \vdash \sum_{x:A} E(x) : type}$$



Π-tribes

Definition

We shall sat that a tribe $\mathcal{C}$ is Π-**closed**, or that it is a Π-**tribe**, if every fibration $E \to A$ has a product along every fibration $f : A \to B$,

$$\begin{array}{ccc} E & & \Pi_f(E) \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

and the structure map $\Pi_f(E) \to B$ is a fibration.

The object $\Pi_f(E)$ is a **product** of $E = (E, p)$ along f. Formally,

$$\Pi_f(E)(b) = \prod_{f(a)=b} E(a)$$

for every $b \in B$.

Π-tribes (2)

It follows that the base change functor $f^\star : \mathcal{C}(B) \to \mathcal{C}(A)$ has a right adjoint

$$f_\star = \Pi_f : \mathcal{C}(A) \to \mathcal{C}(B)$$

for every fibration $f : A \to B$.

For very $X \in \mathcal{C}(A)$ and $Y \in \mathcal{C}(B)$, there is a natural bijection between

$$\frac{\text{the maps} \quad Y \to \Pi_f(X) \quad \text{in } \mathcal{C}(B)}{\text{and the maps} \quad f^\star(Y) \to X \quad \text{in } \mathcal{C}(A)}$$

Product formation

$$\frac{\Gamma, \; x : A \vdash E(x) : \mathit{type}}{\Gamma \vdash \prod_{x:A} E(x) : \mathit{type}}$$

$$\begin{array}{ccc} E & & \Pi_{p_2} E \\ \downarrow & & \downarrow \\ \Gamma \times A & \xrightarrow{p_2} & \Gamma \end{array}$$

Π-tribes (3)

If $\mathcal{C}$ is a Π-tribe, then so is the tribe $\mathcal{C}(A)$ for every $A \in \mathcal{C}$.

A Π-tribe is cartesian closed:

$$B^A = \Pi_A B = \prod_{a:A} B$$

The category $\mathcal{C}(A)$ is cartesian closed for every $A \in \mathcal{C}$.

Examples of Π-tribes

- Every locally cartesian closed category is a Π-tribe.
- The category of small groupoids **Grpd** is a Π-tribe, where a fibration is a Grothendieck fibration.
- The category of Kan complexes is a Π-tribe, where a fibration is a Kan fibration.

Morphisms of Π-tribes

Definition

A **morphism of Π-tribes** $F : \mathcal{C} \to \mathcal{D}$ is a functor which preserves

- terminal objects, fibrations and base changes of fibrations;
- the internal product $\Pi_f(X)$.

The base change functor $u^\star : \mathcal{C}(B) \to \mathcal{C}(A)$ is a morphism of Π-tribes for any map $u : A \to B$ in a Π-tribe $\mathcal{C}$.

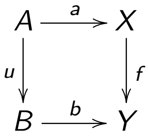
The Yoneda functor $y : \mathcal{C} \to \hat{\mathcal{C}} = [\mathcal{C}^{op}, \mathbf{Set}]$ is a morphism of Π-tribes for any Π-tribe $\mathcal{C}$.

Homotopical logic

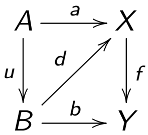
- Weak factorization systems
- Quillen model categories
- Pre-typoi
- Typoi
- Univalent typoi

Weak factorisation systems(1)

The relation $u \pitchfork f$ for two maps $u : A \to B$ and $f : X \to Y$ in a category $\mathcal{C}$ means that every commutative square



has a diagonal filler $d : B \to X$, $du = a$ and $fd = b$.



The map u is said to have the **left lifting property** with f, and the map f to have the **right lifting property with respect to** u.

Weak factorisation systems(2)

For a class of maps $\mathcal{S} \subseteq \mathcal{C}$, let us put

$$\mathcal{S}^{\pitchfork} = \{f \in \mathcal{C} : \forall u \in \mathcal{S} \quad u \pitchfork f\}$$

$$^{\pitchfork}\mathcal{S} = \{u \in \mathcal{C} : \forall f \in \mathcal{S} \quad u \pitchfork f\}$$

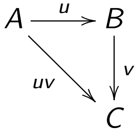
Definition

A pair $(\mathcal{L}, \mathcal{R})$ of classes of maps in a category $\mathcal{C}$ is said to be a **weak factorization system** if the following two conditions are satisfied

- $\mathcal{R} = \mathcal{L}^{\pitchfork}$ and $\mathcal{L} = {}^{\pitchfork}\mathcal{R}$
- every map $f : A \to B$ in $\mathcal{C}$ admits a factorization $f = pu : A \to E \to B$ with $u \in \mathcal{L}$ and $p \in \mathcal{R}$.

Homotopical algebra(1)

Recall that a class $\mathcal{W}$ of maps in a category $\mathcal{E}$ is said to have the **3-for-2 property** (3 apples for the price of two!) if two sides a commutative triangle



belongs to $\mathcal{W}$, then so is the third.

Homotopical algebra(2)

Quillen (1967)

Definition

A **model structure** on a category $\mathcal{E}$ consists on three class of maps $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ respectively called the *cofibrations*, the *weak equivalences* and the *fibrations*, such that :

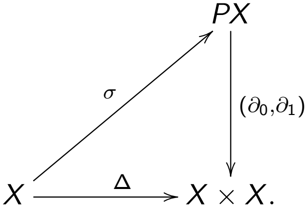
- $\mathcal{W}$ has the 3-for-2 property;
- the pair $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is a weak factorisation system;
- the pair $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is a weak factorisation system.

A **model category** is a category equipped with a model structure.

A map in $\mathcal{W}$ is said to be **acyclic**.

Path object

A **path object** for a fibrant object X in a model category $\mathcal{E}$ is a factorisation of the diagonal $\Delta : X \to X \times X$ as a weak equivalence $\sigma : X \to PX$ followed by a fibration $(\partial_0, \partial_1) : PX \to X \times X$,



The path object is **perfect** if σ is an acyclic cofibration.

Identity type

For every type A there is another type

$$x : A, y : A \ \vdash Id_A(x, y) : type$$

called the **identity type** of A and a term

$$x : A \ \vdash r(x) : Id_A(x, x)$$

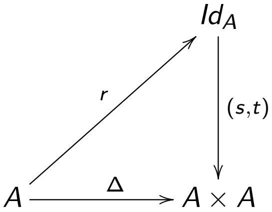
called the **reflexivity term**.

A term $p : Id_A(a, b)$ is a **proof** that $a = b$.

The term $r(x) : Id_A(x, x)$ is the proof that $x = x$.

Identity type(2)

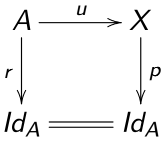
Equivalently, for every $A \in \mathcal{C}$ there is a diagram



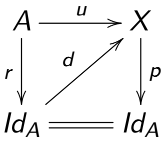
with $(s, t) \in \mathcal{F}$.

The J-rule of type theory

If $p : X \to Id_A$ is a fibration, then every commutative square



has a diagonal filler $d = J(u)$,

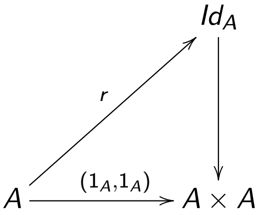


Homotopical algebra and type theory(1)

Theorem (Awodey-Warren):

Martin-Löf type theory can be interpreted in a model category:

- types are interpreted as fibrant objects;
- display maps are interpreted as fibrations;
- the identity type $Id_A \to A \times A$ is a path object for A;
- the reflexivity term $r : A \to Id_A$ is an acyclic cofibration.



Homotopical algebra and type theory(2)

Let $\mathcal{C}(\mathbb{T})$ be the syntactic category of Martin-Löf type theory.

Let $\mathcal{F}$ be the class of display maps in $\mathcal{C}(\mathbb{T})$.

Theorem (Gambino-Garner):

Every map $f : A \to B$ in $\mathcal{C}(\mathbb{T})$ admits a factorization
$f = pu : A \to E \to B$ with $u \in {}^{\pitchfork}\mathcal{F}$ and $p \in \mathcal{F}$.

Pre-typoi

We say that a map in a tribe $\mathcal{C} = (\mathcal{C}, \mathcal{F})$ is **anodyne** if it belongs to the class ${}^{\pitchfork}\mathcal{F}$.

Definition

We say that a tribe $\mathcal{C}$ is a **pre-typos*** if the following two conditions are satisfied

- the base change of an anodyne map along a fibration is anodyne;
- every map $f : A \to B$ admits a factorization $f = pu : A \to E \to B$ with u an anodyne map and p a fibration.

($\star$) Named after a joke by Steve Awodey. Do you have a better name?

Pre-typoi(2)

Examples

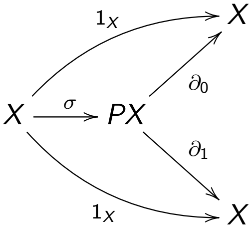
- The category **Grpd**;
- The category of Kan complexes;
- The syntactic category of type theory.

Path objects in a pre-typos

If X is an object of a typos $\mathcal{C}$, then a **perfect* path object** for X is a factorisation

$$\langle \partial_0, \partial_1 \rangle \sigma : X \to PX \to X \times X$$

of the diagonal $X \to X \times X$ as an *anodyne map* $\sigma : X \to PX$ followed by a fibration $\langle \partial_0, \partial_1 \rangle : PX \to X \times X$.



(*) The general notion of path objects will be introduced later.

Paths and equality

The map $\langle \partial_0, \partial_1 \rangle : PX \to X \times X$ of a path object for X is a fibration. Its fiber $PX(x, y)$ at $(x, y) \in X \times X$ is the object of paths $p : x \rightsquigarrow y$. We may write

$$\Gamma \vdash h : f \rightsquigarrow g$$

to indicate that $h : \Gamma \to PX$ is a homotopy between two maps $f, g : \Gamma \to X$.

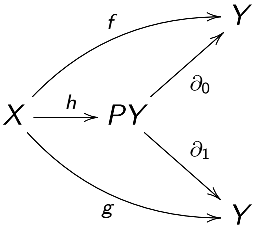
Type theorists write instead

$$\Gamma \vdash h : Id_X(f, g)$$

and regard h as a proof that $f = g$. Weird?

Homotopy relation

A **homotopy** between two maps $f, g : X \to Y$ in a typos $\mathcal{C}$ is a map $h : X \to PY$ such that $\partial_0 h = f$ and $\partial_1 h = g$,



We write $H : f \rightsquigarrow g$ or $f \sim g$.

Homotopy equivalences

Theorem

The homotopy relation $f \sim g$ is a congruence on the arrows of the category $\mathcal{C}$.

The **homotopy category** $Ho(\mathcal{C})$ is the quotient category $\mathcal{C}/\sim$.

A map $f : X \to Y$ in $\mathcal{C}$ is a **homotopy equivalence** if it is invertible in $Ho(\mathcal{C})$.

For example, every anodyne map is a homotopy equivalence.

An object $X \in \mathcal{C}$ is **contractible** if the map $X \to \top$ is a homotopy equivalence.

General path objects

If X is an object of a typos $\mathcal{C}$, then a (general) **path object** for X is a factorisation

$$\langle \partial_0, \partial_1 \rangle \sigma : X \to PX \to X \times X$$

of the diagonal $X \to X \times X$ as a *homotopy equivalence* σ followed by a fibration $\langle \partial_0, \partial_1 \rangle$.

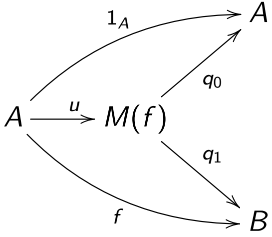
The path object is **perfect** if $\sigma : X \to PX$ is anodyne.

Mapping path object

A **mapping path object** of a map $f : A \to B$ is a factorisation

$$\langle q_0, q_1 \rangle u : A \to M(f) \to A \times B$$

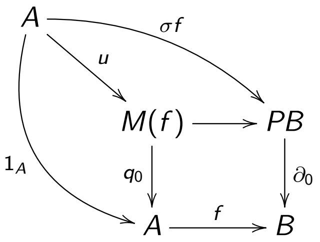
of the map $\langle 1_A, f \rangle : A \to A \times B$ as a homotopy equivalence u followed by a fibration $\langle q_0, q_1 \rangle$,



The mapping path object is **perfect** if u is anodyne.

Homotopy fiber

A mapping path object of a map $f : A \to B$ can be constructed by the following diagram with a pull-back square



Thus, $M(f) = \{(p, x, y) |\ x : A,\ y : B,\ p : f(x) \rightsquigarrow y\}$. The fiber of the projection $M(f) \to B$ at $b \in B$ is the **homotopy fiber** of f.

$$M(f)(b) = \{(p, x) |\ x : A,\ p : f(x) \rightsquigarrow y\}$$

n-types

Let $\mathcal{C}$ be a pre-typos. If $X \in \mathcal{C}$, then the fibration $\langle \partial_0, \partial_1 \rangle : PX \to X \times X$ is an object $P(X)$ of $\mathcal{C}(X \times X)$.

Definition

We say that an object $X \in \mathcal{C}$ is a (-1)-**type** if $P(X)$ is contractible in the pre-typos $\mathcal{C}(X \times X)$.

An object $X \in \mathcal{C}$ is a (-1)-type if and only if the map $X \to X \times X$ is a homotopy equivalence.

A (-1)-type is like a *truth value*.

n-types

Definition

If $n \geq 0$, then an object $X \in \mathcal{C}$ is said to be a n-**type** if $P(X)$ is a $(n-1)$-type in $\mathcal{C}(X \times X)$.

A 0-type is like a *set*.

A 1-type is like a *groupoid*.

A 1-type is like a *2-groupoid*.

Morphisms of pre-typoi

Definition

A **morphism of pre-typoi** $F : \mathcal{C} \to \mathcal{D}$ is a functor which preserves

- terminal objects, fibrations and base changes of fibrations;
- the homotopy relation.

For example, the base change functor $u^\star : \mathcal{C}(B) \to \mathcal{C}(A)$ is a morphism of pre-typoi for any map $u : A \to B$ of a pre-typos $\mathcal{C}$.

Typoi*

Definition

A pre-typos $\mathcal{C}$ is called a **typos*** if it is a Π-tribe and the product functor $\Pi_f : \mathcal{C}(A) \to \mathcal{C}(B)$ preserves the homotopy relation for every fibration $f : A \to B$.

If $\mathcal{C}$ is a typos, then so is the tribe $\mathcal{C}(A)$ for any object $A \in \mathcal{C}$.

$(\star)$ Do you have a better name?

Examples

Theorem

(Hoffman and Streicher) The category of groupoids **Grpd** *has the structure of a typos in which the fibrations are the Grothendieck fibrations.*

Theorem

(Awodey-Warren-Voevodsky) The category of Kan complexes has the structure of a typos in which the fibrations are the Kan fibrations.

Theorem

(Gambino-Garner) The syntactic category of type theory has the structure of a typos in which the fibrations are constructed from the display maps.

From typoi to hyperdoctrines

If $u : A \to B$ is a map in a typos $\mathcal{C}$, then the functor

$$Ho(u^\star) : Ho(\mathcal{C}(B)) \to Ho(\mathcal{C}(A))$$

has a both a left adjoint and a right adjoint.

The functor

$$A \mapsto Ho(\mathcal{C}(A))$$

is a hyper-doctrine in the sense of Lawvere!

Morphisms of typoi

Definition

A **morphism of typoi** $F : \mathcal{C} \to \mathcal{D}$ is a functor which preserves

- terminal objects, fibrations and base changes of fibrations;
- the internal products $\Pi_f(X)$;
- the homotopy relation.

For example, the base change functor $u^\star : \mathcal{C}(B) \to \mathcal{C}(A)$ is a morphism of typoi for any map $u : A \to B$ in a typos $\mathcal{C}$.

Internal statements

Let A be an object of a typos $\mathcal{C}$.

If $PA \to A \times A$ is a path object for A, then the object

$$T_{-1}(A) = \prod_{x:A} \prod_{y:A} PA(x, y)$$

is the internal statement that A is a (-1)-type.

A term $p : T_{-1}(A)$ is a proof that A is a (-1)-type.

Internal statements(2)

The object

$$Cont(A) = A \times T_{-1}(A)$$

is the internal statement that A is contractible.

A term $p : A \times T_{-1}(A)$ is a proof that A is contractible.

If $n \geq 0$, then the object

$$T_n(A) = \prod_{x:A} \prod_{y:A} T_{n-1}(PA(x,y))$$

is the internal statement that A is a n-type.

A term $p : T_n(A)$ is a proof that A is a n-type.

Internal equivalences

A fibration $p : X \to B$ is a homotopy equivalence if and only $\Pi_B(X, p)$ is contractible.

Thus, $Cont(\Pi_B(X, p))$ is the internal statement that $p : X \to A$ is a homotopy equivalence.

A general map $f : A \to B$ is a homotopy equivalence if and only if the fibration $q_1 : M(f) \to B$ is a homotopy equivalence.

Thus, $Cont(\Pi_B(M(f), q_1))$ is the internal statement that $f : A \to B$ is a homotopy equivalence.

Classifying equivalences

For any pair of objects X and Y of a typos, there is an object $Eq(X, Y)$ classifying the homotopy equivalences $X \to Y$.

For every fibration $X \to A$, there is a category object

$$(s, t) : Eq_A(X) \to A \times A$$

where

$$Eq_A(X)(a, b) = Eq(X(a), X(b))$$

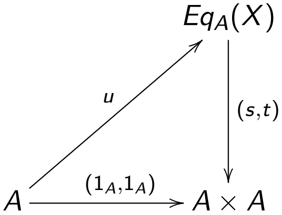
for $a : A$ and $b : A$.

Univalent fibrations

Definition

We say that a fibration $X \to A$ is **univalent** if the unit map $u : A \to Eq_A(X)$ is an equivalence.

A fibration $X \to A$ is univalent if and only if the factorization



is a path object for A.

Small fibrations and universes

A typos $\mathcal{C}$ may contain a sub-typos of **small fibrations**.

A small fibration $q : U' \to U$ is **universal** if for every small fibration $p : X \to A$ there exists a cartesian square:

$$\begin{array}{ccc} X & \xrightarrow{\chi'} & U' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{\chi} & U. \end{array}$$

The map χ is **classifying** (X, p).

A **universe** is the codomain of a universal small fibration $U' \to U$.

Martin-Löf axiom: *There is a universe* U.

Univalent typoi

We would like to say that the pair (χ, χ') classifying a fibration $p : X \to A$ is homotopy unique.

Voevodsky axiom: *The universal fibration $U' \to U$ is univalent.*

Theorem (Voevodsky)

The category of Kan complexes **Kan** has the structure of a univalent typos in which the fibrations are the Kan fibrations.

Conclusions

Homotopy type theory is soluble in category theory

Bibliography

THE BOOK OF INFORMAL TYPE THEORY