Additive ∞-categories and canonical monoidal structures II

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joint work with David Gepner and Moritz Groth

Recall from Moritz's talk

<u>Recall:</u> $\mathcal{C}$ presentable ∞-category ($= (\infty, 1)$-category)

- $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C})$: ∞-category of commutative monoids in $\mathcal{C}$
- $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C})$: ∞-category of commutative groups in $\mathcal{C}$
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Examples

$\mathcal{C}$	$\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C})$	$\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C})$	$\mathrm{Sp}(\mathcal{C})$
Set	abelian monoids	abelian groups	trivial
$\mathcal{C}at$	$\mathcal{S}ym\mathcal{M}on\mathcal{C}at$	Picard groupoids	trivial
$\mathcal{C}at_\infty$	$\mathcal{S}ym\mathcal{M}on\mathcal{C}at_\infty$	Picard ∞-groupoids	Spectra
$\mathcal{S}$ = Spaces	$\mathbb{E}_\infty$-spaces	grouplike $\mathbb{E}_\infty$-spaces	Spectra

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$\mathcal{C}$ presentable $\Rightarrow$ $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C})$, $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C})$ and $\mathrm{Sp}(\mathcal{C})$ are presentable

Universal Property

Theorem (Gepner, Groth, N.)

- $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C})$ *is universal preadditive* ∞*-category on* $\mathcal{C}$.
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$$\mathcal{P}\mathrm{r}^{\mathrm{L}} := \left\{ \begin{matrix} \infty\text{-category of presentable } \infty\text{-categories and} \\ \text{left adjoint functors.} \end{matrix} \right\}$$

(Bousfield-)localizations

$$\mathrm{Mon}_{\mathbb{E}_\infty}, \mathrm{Grp}_{\mathbb{E}_\infty}, \mathrm{Sp} : \qquad \mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$$

Local objects: (pre)additive / stable ∞-categories.

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Corollary

There are canonical left adjoint functors

$$\mathcal{C} \to \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C}) \to \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C}).$$

Main theorem

Theorem (Gepner, Groth, N.)

$\mathcal{C}$ *closed symmetric monoidal, presentable* ∞*-category*

1. $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C}), \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C})$ *and* $\mathrm{Sp}(\mathcal{C})$ *admit symmetric monoidal structures:*
    - *Tensor products preseves colimits in both variables.*
    - *Free functors* $\mathcal{C} \to \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C})$, $\mathcal{C} \to \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C})$ *and* $\mathcal{C} \to \mathrm{Sp}(\mathcal{C})$ *admit symmetric monoidal structures.*

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2. *These symmetric monoidal structures are (essentially) unique.*
3. *The functors* $\mathcal{C} \to \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C}) \to \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C})$ *admit unique structures of symmetric monoidal functors.*

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Assume $\mathcal{C}$ cartesian closed

$$\mathcal{R}ig_{\mathbb{E}_k}(\mathcal{C}) := \mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C})^{\otimes}) \qquad \text{'semirings in } \mathcal{C}\text{'}$$

$$\mathcal{R}ing_{\mathbb{E}_k}(\mathcal{C}) := \mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C})^{\otimes}) \qquad \text{'rings in } \mathcal{C}\text{'}$$

Examples of tensor product I

Case $\mathcal{C} = \mathrm{Set}$: Ordinary tensor product on abelian monoids/groups.

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(first constructed by Hermida, Power,
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Proposition (Recognition principle)

Let $(C, \otimes)$ be a $\mathbb{E}_k$-monoidal (∞)-category such that

1. *C has coproducts (denoted $\oplus$).*
2. *$\otimes : C \times C \to C$ preserves coproducts in both variables.*

Then $(C, \oplus, \otimes)$ is canonically a $\mathbb{E}_k$-semiring (∞)-category.

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- commutative ring R
  $\to (\mathrm{Mod}_R, \oplus, \otimes)$ is a commutative semiring category
- $\mathbb{E}_k$-ring spectrum R
  $\to (\mathrm{Mod}_R, \oplus, \otimes)$ is a $\mathbb{E}_{k-1}$-semiring ∞-category

Examples of tensor product II

Case $\mathcal{C} = \text{Spaces}$: tensor product on (grouplike) $\mathbb{E}_\infty$-spaces and spectra. Functor

$$\mathbb{E}_\infty\text{-spaces} \to \mathrm{Sp}$$

is symmetric monoidal (in a unique way)

'Canonical multiplicative delooping machine'.

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Direct sum K-theory functor

$$K : \mathcal{S}ym\mathcal{M}on\mathcal{C}at \xrightarrow{(-)^\sim} \mathcal{S}ym\mathcal{M}on\mathcal{C}at \xrightarrow{|-|} \mathbb{E}_\infty\text{-spaces} \to \mathrm{Sp}$$

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- commutative ring R
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- connective $\mathbb{E}_k$-ringspectrum R
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Proof I

Ingredients for proof of main Theorem:

1. Localization property:
   $\mathrm{Mon}_{\mathbb{E}_\infty}, \mathrm{Grp}_{\mathbb{E}_\infty}, \mathrm{Sp} : \mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ are localizations
2. Basechange property:

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$$\mathcal{P}\mathrm{r}^{\mathrm{L}} = \Big\{\text{presentable } \infty\text{-categories and left adjoint functors}\Big\}$$

Lurie: tensor product $\otimes : \mathcal{P}\mathrm{r}^{\mathrm{L}} \times \mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$.

- universal property: $\mathrm{Fun}^L(A \otimes B, C) \subset \mathrm{Fun}(A \times B, C)$
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- explicit formula $A \otimes B = \mathrm{Fun}^R(A^{op}, B)$.
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Holds more generally (algebraic theories, operads...):

$\mathbb{T}$ Lawvere algebraic theory $\Rightarrow \mathrm{Mod}_{\mathbb{T}}(\mathcal{C} \otimes \mathcal{D}) \simeq \mathrm{Mod}_{\mathbb{T}}(\mathcal{C}) \otimes \mathcal{D}$

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Definition

A localization $L : \mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ is called *smashing* if $L(\mathcal{C}) \simeq \mathcal{C} \otimes \mathcal{M}$ for $\mathcal{M} \in \mathcal{P}\mathrm{r}^{\mathrm{L}}$.

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- Necessarily $\mathcal{M} \cong L(\mathcal{S})$.
- $\mathcal{M}$ is idempotent monoid
- {smashing localizations} $\overset{1-1}{\longleftrightarrow}$ {idempotent monoids}

Proof III

Proposition

$L : \mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ smashing localization, $\mathcal{C} \in \mathcal{P}\mathrm{r}^{\mathrm{L}}$ closed symmetric monoidal.

1. *Functor L is symmetric monoidal*

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1. *Functor L is symmetric monoidal*
2. *$L\mathcal{C}$ admits closed symmetric monoidal structure.*
   Unique s.t. $\mathcal{C} \to L\mathcal{C}$ admits symmetric monoidal structure.
3. *For $\mathcal{D}$ local object $\mathrm{Fun}^{L,\otimes}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}^{L,\otimes}(L\mathcal{C}, \mathcal{D})$.*

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L' second smashing localization with $L' < L$:

$L\mathcal{C} \to L'\mathcal{C}$ admits unique symmetric monoidal structure.

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Proof of theorem.

Apply proposition to smashing localizations

$$\mathrm{Mon}_{\mathbb{E}_\infty}, \mathrm{Grp}_{\mathbb{E}_\infty}, \mathrm{Sp} : \mathcal{P}r^{\mathrm{L}} \to \mathcal{P}r^{\mathrm{L}}$$

□

Summary and Outlook

1. $\mathcal{C}$ presentable $\Rightarrow \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C}), \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C}), \mathrm{Sp}(\mathcal{C})$ presentable
2. Smashing localizations $\mathrm{Mon}_{\mathbb{E}_\infty}, \mathrm{Grp}_{\mathbb{E}_\infty}, \mathrm{Sp} : \mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$
   local objects (pre)addtive/stable ∞-categories.
   $\Rightarrow$ universal properties
3. unique tensor product on $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C}), \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C})$ and $\mathrm{Sp}(\mathcal{C})$.
   $\Rightarrow$ tensor product on $\mathcal{S}ym\mathcal{M}on\mathcal{C}at$, $\mathbb{E}_\infty$-spaces...
4. Unique monoidal functors $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C}) \to \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C})$
   $\Rightarrow$ multiplicative infinite loopspace machine
5. K-theory functor $K : \mathcal{C}at_\infty \to \mathrm{Sp}$ lax symmetric monoidal

6. $F : \mathcal{C} \to \mathcal{D}$ product preserving
   $\Rightarrow \underline{F} : \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C}) \to \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{D})$ lax monoidal
   F left adjoint $\Rightarrow \underline{F}$ symmetric monoidal
7. $\exists$ Algebraic theories $\mathbb{T}_k, \mathbb{T}'_k$ s.t.

$$\mathcal{R}ig_{\mathbb{E}_k}(\mathcal{C}) \simeq \mathrm{Mod}_{\mathbb{T}_k}(\mathcal{C}) \quad \text{and} \quad \mathcal{R}ing_{\mathbb{E}_k}(\mathcal{C}) \simeq \mathrm{Mod}_{\mathbb{T}'_k}(\mathcal{C})$$