

The “Triple Category” of Bicategories

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Introduction

- ▶ Common wisdom: The study of 2-dimensional structures leads inexorably to 3-dimensional ones.
- ▶ Consider the theory of bicategories: What 3-dimensional structure do they present?
- ▶ No matter how it's set up, something doesn't work!
- ▶ Grandis and I looked at weak triple categories. We found no full-fledged natural examples!
- ▶ Now think this is a basic fact about triple categories. (Like the symmetry isomorphisms for monoidal categories.)

Bicategories: What's there (Bénabou)

- ▶ 0-cells - Bicategories $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$
- ▶ 1-cells
 - ▶ Lax functors $F : \mathcal{A} \longrightarrow \mathcal{B}$

$$\begin{array}{ccc} A & \begin{array}{c} \xrightarrow{f} \\ \alpha \Downarrow \\ \xrightarrow{g} \end{array} & A' \end{array} \mapsto \begin{array}{ccc} FA & \begin{array}{c} \xrightarrow{Ff} \\ F\alpha \Downarrow \\ \xrightarrow{Fg} \end{array} & FA' \end{array}$$

- ▶ Functorial on the α 's (vertical composition)
- ▶ Horizontally lax

$$\begin{array}{ccccc} & & FA' & & \\ & Ff \nearrow & \Downarrow & \searrow Ff' & \\ FA & \xrightarrow{\quad} & FA'' & & \\ & F(f'f) & & & \end{array}$$

- ▶ Satisfy coherence conditions
- ▶ Horizontal composition of 2-cells is preserved as much as it can be
- ▶ Occur in practice
- ▶ Compose, strictly associative and unitary

What's there: 2-cells

Lax transformations

$$\begin{array}{ccc}
 & \mathcal{A} & \\
 & \begin{array}{c} \xrightarrow{F} \\ t \Downarrow \\ \xrightarrow{G} \end{array} & \mathcal{B} \\
 & & \\
 A \xrightarrow{f} A' \mapsto & \begin{array}{ccc} FA & \xrightarrow{Ff} & FA' \\ \downarrow tA & \xRightarrow{tf} & \downarrow tA' \\ GA & \xrightarrow{Gf} & GA' \end{array} &
 \end{array}$$

- ▶ t respects composition of 1-cells
- ▶ t is natural with respect to 2-cells
- ▶ Occur in practice

- ▶ Compose vertically

$$\begin{array}{ccc}
 & \mathcal{A} & \\
 & \begin{array}{c} \xrightarrow{t \Downarrow} \\ \xrightarrow{u \Downarrow} \end{array} & \mathcal{B} \\
 & &
 \end{array}$$

associative and unitary up to “canonical” isomorphisms (3-cells)

What's there: Other 2-cells

- ▶ Oplax transformations

$$A \xrightarrow{f} A' \quad \mapsto \quad \begin{array}{ccc} FA & \xrightarrow{Ff} & FA' \\ \downarrow tA & \Downarrow tf & \downarrow tA' \\ GA & \xrightarrow{Gf} & GA' \end{array}$$

- ▶ Also occur in practice
- ▶ No hint as to which should be called lax or oplax
- ▶ There are also strong transformations

What's there: 3-cells

- Modifications

$$\begin{array}{ccc} & t & \\ F & \xrightarrow{\quad} & G \\ & \tau \Downarrow & \\ & t' & \end{array}$$

- For every A

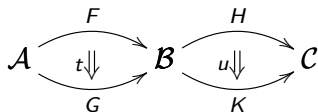
$$\begin{array}{ccc} & tA & \\ FA & \xrightarrow{\quad} & GA \\ & \tau A \Downarrow & \\ & t'A & \end{array}$$

- For every $f : A \multimap A'$

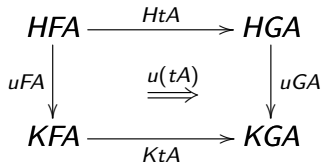
$$\begin{array}{ccc} Gf \cdot tA & \xrightarrow{tf} & tA' \cdot Ff \\ \downarrow Gf \cdot \tau A & & \downarrow \tau A' \cdot Ff \\ Gf \cdot t'A & \xrightarrow{t'f} & t'A' \cdot Ff \end{array}$$

- Lax functors, lax transformations, modifications yield a bicategory $\mathbb{Lax}(\mathcal{A}, \mathcal{B})$

What's not there: Horizontal composition



- For A there are two choices for $(ut)A : HFA \longrightarrow KGA$



- Neither choice is associative: e.g. if $v : L \longrightarrow M$ is another transformation, and we use the top composite

$$(v(ut))A = vKGA \cdot L(uGA \cdot HtA)$$

$$((vu)t)A = vKGA \cdot LuGA \cdot LHtA$$

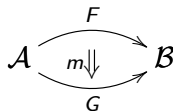
Horizontal composition - continued

- Worse - neither choice will produce a laxity cell $ut(f)$
- Even whiskering doesn't

$$\mathcal{A} \begin{array}{c} \xrightarrow{F} \\ t \Downarrow \\ \xrightarrow{G} \end{array} \mathcal{B} \xrightarrow{H} \mathcal{C}$$

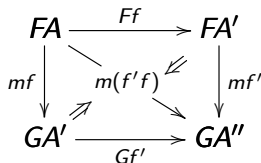
$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FA' \\ \downarrow tA & \Rightarrow^{tf} & \downarrow tA' \\ GA & \xrightarrow{Gf} & GA' \end{array} \quad \mapsto \quad \begin{array}{ccc} HFA & \xrightarrow{\quad} & HFA' \\ \downarrow & \Rightarrow & \downarrow \\ HGA & \xrightarrow{\quad} & HGA' \end{array}$$

Modules [CKSW]



► $m : A \xrightarrow{f} A' \mapsto mf : FA \longrightarrow GA'$

► $A \xrightarrow{f} A' \xrightarrow{f'} A''$



► $\rho_m : mf' \cdot Ff \longrightarrow m(f'f)$

► $\lambda_m : Gf' \cdot mf \longrightarrow m(f'f)$

► m is functorial on 2-cells

► ρ_m and λ_m are natural

► ρ_m and λ_m are associative (3 equations)

► ρ_m , λ_m are unitary (2 equations)

Modules: Examples

- ▶ $\mathbf{Lax} \quad \mathbf{1} \longrightarrow \mathbf{Span}$ is a category
- ▶ Module is a profunctor
- ▶ Functors are *not* lax transformations
They are oplax transformations for which tA is a *function*
- ▶ More generally $\mathbf{Lax} \quad \mathbf{1} \longrightarrow \mathbf{V}\text{-}\mathcal{M}\text{at}$ is a $\mathbf{V}$ -category
Module is a $\mathbf{V}$ -profunctor

Modules: Lax transformations

- ▶ A lax transformation $t : F \longrightarrow G$ gives a module $\bar{t} : F \longrightarrow G$

$$\text{For } f : A \longrightarrow A', \quad \bar{t}f = FA \xrightarrow{Ff} FA' \xrightarrow{tA'} FA'$$

$$\rho_{\bar{t}} : \bar{t}f' \cdot Ff \longrightarrow \bar{t}(f'f)$$

$$tA'' \cdot Ff' \cdot Ff \xrightarrow{tA'' \cdot \phi} tA'' \cdot F(f'f)$$

$$\lambda_{\bar{t}} : Gf' \cdot \bar{t}f \longrightarrow \bar{t}(f'f)$$

$$Gf' \cdot tA' \cdot Ff \xrightarrow{tf' \cdot Ff} tA'' \cdot Ff' \cdot Ff \xrightarrow{tA'' \cdot \phi} tA'' \cdot F(f'f)$$

- ▶ An oplax transformation $t : F \longrightarrow G$ also produces a module $\tilde{t} : F \longrightarrow G$

$$\tilde{t}f = FA \xrightarrow{tA} GA \xrightarrow{Gf} GA'$$

Modules: Horizontal composition

$$\begin{array}{ccccc}
 & F & & H & \\
 \mathcal{A} & \xrightarrow{\quad} & \mathcal{B} & \xrightarrow{\quad} & \mathcal{C} \\
 & m \Downarrow & & n \Downarrow & \\
 & G & & K &
 \end{array}$$

$$(A \xrightarrow{f} A') \quad \mapsto^m \quad (FA \xrightarrow{mf} GA') \quad \mapsto^n \quad (HFA \xrightarrow{n(mf)} KGA')$$

$$(nm)(f) = n(mf)$$

$$\rho_{nm} : (nm)f' \cdot HFf \longrightarrow (nm)(f'f)$$

$$n(mf') \cdot HFf \xrightarrow{\rho_n} n(mf' \cdot Ff) \xrightarrow{n(\rho_m)} n(m(f'f))$$

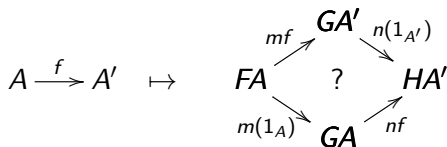
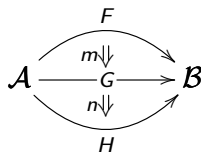
$$\lambda_{nm} : KGf' \cdot (nm)f \longrightarrow (nm)(f'f)$$

$$KGf' \cdot n(mf) \xrightarrow{\lambda_n} n(Gf' \cdot mf) \xrightarrow{n(\lambda_m)} n(m(f'f))$$

- Strictly associative and unitary

Modules: Vertical composition

Does not usually exist!



Theorem (CKSW)

If $\mathcal{B}$ is locally cocomplete and $\mathcal{A}$ small, then the composite $n \cdot m$ exists, and is associative and unitary up to coherent isomorphism.

Proof.

- ▶ $(n \cdot m)(f) = \varinjlim_{\alpha} (nf_2 \cdot mf_1) \quad \alpha : f_2 f_1 \longrightarrow f.$
- ▶ There is an unbiased triple composite

$$(p \cdot n \cdot m)(f) = \varinjlim_{\alpha} (pf_3 \cdot nf_2 \cdot mf_1)$$

taken over all $\alpha : f_3 f_2 f_1 \longrightarrow f$



$$\begin{aligned} ((p \cdot n) \cdot m)(f) &\simeq \varinjlim (p \cdot n)(f_2) \cdot mf_1 \\ &\simeq \varinjlim (\varinjlim f_{22} \cdot nf_{21}) \cdot mf_1 \\ &\simeq \varinjlim (pf_{22} \cdot nf_{21} \cdot mf_1) \end{aligned}$$

where the last colimit is taken over all pairs $\alpha_1 : f_2 f_1 \longrightarrow f$ and

$$\alpha_2 : f_{22} f_{21} \longrightarrow f_2$$

The diagram illustrates the commutativity of the triple composite. It shows four objects: A_1 (top left), A_2 (top right), A (bottom left), and A' (bottom right). The arrows are: $A_1 \xrightarrow{f_{21}} A_2$, $A_1 \xrightarrow{f_1} A$, $A_1 \xrightarrow{f_2} A'$, $A \xrightarrow{f} A'$, and $A_2 \xrightarrow{f_{22}} A'$. Double arrows with labels indicate the following commutative squares: $A_1 \xrightarrow{f_1} A \xrightarrow{f} A'$ (labeled α_1), $A_1 \xrightarrow{f_2} A' \xrightarrow{f_{22}} A'$ (labeled α_2), and $A_1 \xrightarrow{f_2} A' \xrightarrow{f_{22}} A'$ (labeled α_2).

Theorem (CKSW)

If n is of the form $\bar{t}$ for a lax transformation t , then $n \cdot m$ exists. Similarly if m is of the form $\tilde{t}$ for an oplax transformation then $n \cdot m$ exists.

- ▶ $(\bar{t} \cdot m)(f) = (FA \xrightarrow{mf} GA' \xrightarrow{tA'} HA')$
 $(n \cdot \tilde{t})(f) = (FA \xrightarrow{tA} GA \xrightarrow{nf} HA')$
- ▶ $\bar{t}' \cdot \bar{t} = \overline{t' \cdot t}$ and $\tilde{t}' \cdot \tilde{t} = \widetilde{t' \cdot t}$

Three-dimensional structure: Modulations [CKSW]

There are morphisms of modules, 3-cells for $Bicat$, called *modulations*

$$\mu : m \longrightarrow n : F \longrightarrow G$$



$$f : A \longrightarrow A' \quad \mapsto \quad \begin{array}{ccc} & \xrightarrow{mf} & \\ FA & \mu f \Downarrow & GA' \\ & \xrightarrow{nf} & \end{array}$$

- Natural in f
- Equivariant

Three-dimensional structure: Multimodulations

Bimodulation

$$\begin{array}{ccc} & F_1 & \\ m_1 \nearrow & & \searrow m_2 \\ F_0 & \xrightarrow{n} & F_2 \\ & \mu \Downarrow & \end{array}$$



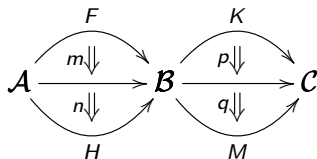
$$(A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2) \quad \mapsto \quad \begin{array}{ccc} & F_1 A_1 & \\ m_1 f_1 \nearrow & & \searrow m_2 f_2 \\ F_0 A_0 & \xrightarrow{n(f_2 f_1)} & F_2 A_2 \\ & \Downarrow \mu(f_2, f_1) & \end{array}$$

- μ is natural in f_1 and f_2
- “Equivariant”

They determine the composite of m_1 and m_2

- Composition becomes a question of representability

Interchange

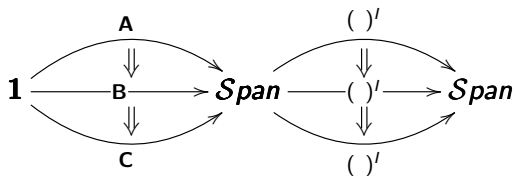


$$(qn) \cdot (pm)(f) = \lim_{\rightarrow \alpha} q(nf_2) \cdot p(mf_1); \quad (\alpha : f_2 f_1 \longrightarrow f)$$

$$\begin{aligned} (q \cdot p)(n \cdot m)(f) &= \lim_{\rightarrow \alpha} q((n \cdot m)f) \cdot p((n \cdot m)f) \\ &= \lim_{\rightarrow \alpha} q(\lim_{\rightarrow \alpha_2} (nf_{22} \cdot mf_{21})) \cdot p(\lim_{\rightarrow \alpha_1} (nf_{12} \cdot mf_{11})) \\ &\quad (\alpha_1 : f_{12} f_{11} \longrightarrow f_1 \quad \alpha_2 : f_{22} f_{21} \longrightarrow f_2) \end{aligned}$$

- ▶ No reason for p or q to preserve $\lim_{\rightarrow}$
- ▶ p and q don't preserve composition

Example



$P : \mathbf{A} \multimap \mathbf{B}$, $Q : \mathbf{B} \multimap \mathbf{C}$ profunctors

$()^I \multimap ()^I$ identity modules

$P^I : \mathbf{A}^I \multimap \mathbf{B}^I$

$$P^I(\langle A_i \rangle, \langle B_i \rangle) = \prod_i P(A_i, B_i)$$

An element is a family of elements $A_i \xrightarrow{x_i} B_i$

$$Q^I \otimes P^I \longrightarrow (Q \otimes P)^I$$

An element on LHS is an equivalence class of pairs of families

$$[\langle A_i \xrightarrow{\bullet x_i} B_i \rangle_i, \langle B_i \xrightarrow{\bullet y_i} C_i \rangle_i]_{\langle B_i \rangle}$$

An element on RHS is a family of equivalence classes

$$\langle [A_i \xrightarrow{\bullet x_i} B_i \xrightarrow{\bullet y_i} C_i] \rangle_i$$

In the first, the equivalence involves a path of families whereas in the second, it's a family of paths which can have unbound length if I is infinite.

Conclusion

- ▶ [CKSW] talk about a smooth three-dimensional structure of bicategories, lax functors, modules and modulations.
- ▶ What is this?
- ▶ Perhaps we can't have vertical and horizontal composition simultaneously.
- ▶ Interchange should be relaxed.