Weakly Globular Double Categories

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Outline

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Bicategories

- We are considering models for a 2-category of weak 2-categories.
- Our first model is the 2-category $\mathbf{Bicat}_{\text{icon}}$ of
  - Objects: Bicategories
  - Arrows: Normal Homomorphisms
  - 2-Cells: Icons (identity component pseudo natural transformations)

Bicategories

- We are considering models for a 2-category of weak 2-categories.
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The 2-Nerve

Definition (Lack-Paoli, 2008)

The 2-nerve of a bicategory, $N(\mathcal{B}) : \Delta^{\mathrm{op}} \to \mathbf{Cat}$:

- 0-simplices $N(\mathcal{B})_0$: the discrete category $\mathcal{B}_0$

$$\bullet X \qquad \bullet Y \qquad\qquad \begin{tikzcd} \bullet X \arrow[loop left, "\mathrm{id}"] \end{tikzcd} \quad \begin{tikzcd} \bullet Y \arrow[loop left, "\mathrm{id}"] \end{tikzcd}$$

- 1-simplices $N(\mathcal{B})_1$:

Objects

$$\begin{tikzcd} X\bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & \bullet Y \end{tikzcd}$$

$$\begin{tikzcd} S\bullet \arrow[r, "h"] & \bullet T \end{tikzcd}$$

Arrows

$$\begin{tikzcd} \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\alpha\Downarrow"] & \bullet \end{tikzcd}$$

- 2-simplices $N(\mathcal{B})_2$ has objects

$$\begin{tikzcd} & \cdot \arrow[dr, "g"] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, shift left=3, "\alpha\cong"] & & \cdot \end{tikzcd}$$

The 2-Nerve [Lack-Paoli, 2008]

- This 2-nerve N is the object part of a fully faithful 2-functor

$$N\colon \mathbf{Bicat}_{\text{icon}} \to [\Delta^{\text{op}}, \mathbf{Cat}]$$

which has a left biadjoint.

- $N(\mathcal{B})$ is a Tamsamani weak 2-category.

Theorem (Lack-Paoli, 2008)

There is a 2-adjoint biequivalence

$$\mathbf{Bicat}_{\text{icon}} \simeq \mathbf{Ta}_2.$$

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Tamsamani Weak 2-Categories

This leads to our second model: **$\mathbf{Ta}_2$**

- Objects:

$$X_* : \Delta^{\mathrm{op}} \to \mathbf{Cat}$$

  such that
  - X_0 is discrete
  - the Segal maps

$$\eta_k : X_k \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

    are equivalences of categories
- Arrows: pseudo natural transformations
- 2-Cells: modifications

Remarks

- $X_* \colon \Delta^{\mathrm{op}} \to \mathbf{Cat}$ is the horizontal nerve of an internal category in **Cat**, i.e., a double category, if and only if the Segal maps are isomorphisms.
- [Paoli-P] For each Tamsamani 2-category

$$X_* \colon \Delta^{\mathrm{op}} \to \mathbf{Cat},$$

there is a functor

$$(RX)_* \colon \Delta^{\mathrm{op}} \to \mathbf{Cat}$$

such that

  - $(RX)_n \simeq X_n$ for all $n \geq 0$
  - $(RX)_k \cong (RX)_1 \times_{(RX)_0} \overset{k}{\cdots} \times_{(RX)_0} (RX)_1$
- Note: We have traded a discrete X_0 and Segal $\simeq$ for a posetal groupoid $(RX)_0$ and Segal $\cong$.

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  - $(RX)_n \simeq X_n$ for all $n \geq 0$
  - $(RX)_k \cong (RX)_1 \times_{(RX)_0} \overset{k}{\cdots} \times_{(RX)_0} (RX)_1$
- Note: We have traded a discrete X_0 and Segal $\simeq$ for a posetal groupoid $(RX)_0$ and Segal $\cong$.

Weakly Globular Double Categories

Definition

A (strict) double category is weakly globular if it has the following properties:

- There is an equivalence of categories
$$\gamma \colon \mathbb{X}_0 \to \mathbb{X}_0^d,$$
where $\mathbb{X}_0$ is the category of vertical arrows and $\mathbb{X}_0^d$ is the discrete category of its path components.
- γ induces an equivalence of categories, for $n \geq 2$,
$$\mathbb{X}_1 \times_{\mathbb{X}_0} \overset{n}{\cdots} \times_{\mathbb{X}_0} \mathbb{X}_1 \simeq \mathbb{X}_1 \times_{\mathbb{X}_0^d} \overset{n}{\cdots} \times_{\mathbb{X}_0^d} \mathbb{X}_1$$

Remark

The second condition is satisfied when at least one of $d_0, d_1 : \mathbb{X}_1 \rightrightarrows \mathbb{X}_0$ is an isofibration.

Weakly Globular Double Categories

This leads to our third model: $\mathbf{WGDbl}_{\mathrm{ps}}$

- Objects: Weakly globular double categories
- Arrows: Pseudo functors (which correspond to pseudo natural transformations between the horizontal nerves)
- 2-Cells: (Pseudo) Vertical transformations

Theorem (Paoli-P)

There is a biequivalence

$$\mathbf{Bicat}_{\mathrm{Icon}} \simeq \mathbf{WGDbl}_{\mathrm{ps}}.$$

Additional Structure

There is a second 2-category structure for weakly globular double categories.

WGDbl$_{\mathrm{st}}$ has

- Arrows: strict functors
- 2-Cells: horizontal transformations

Remark

A 2-equivalence in **WGDbl**$_{\mathrm{ps}}$ does not imply a 2-equivalence in **WGDbl**$_{\mathrm{st}}$ or vice versa. So when considering universal properties we would like to have one for each direction.

The Biequivalence

There are 2-functors

$$\mathbf{Bicat}_{\mathrm{Icon}} \underset{\mathbf{Bic}}{\overset{\mathbf{Dbl}}{\rightleftarrows}} \mathbf{WGDbl}_{\mathrm{ps}}$$

which form a biequivalence of 2-categories, but not an adjunction.

The Functor **Bic**

For a double category $\mathbb{X}$, the fundamental bicategory $\mathbf{Bic}(\mathbb{X})$ is defined by:

- objects $\mathbf{Bic}(\mathbb{X})_0 = \pi_0\mathbb{X}_0$
- A in $\mathbb{X}$ gives rise to $\bar{A}$ in $\mathbf{Bic}(\mathbb{X})$
- arrows

$$Hom_{\mathbf{Bic}\mathbb{X}}(\bar{A}, \bar{B}) = \coprod_{\substack{\bar{A}' = \bar{A} \\ \bar{B}' = \bar{B}}} Hom_{\mathbb{X},h}(A', B').$$

- a 2-cell $\bar{A} \underset{g}{\overset{f}{\Downarrow\alpha}} \bar{B}$ is given by a double cell

$$\begin{tikzcd} A_1 \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\alpha"] & B_1 \arrow[d, "w"] \\ A_2 \arrow[r, "g"'] & B_2 \end{tikzcd} .$$

Horizontal Composition of Arrows

- Recall that there is an equivalence of categories

$$\mathbb{X}_1 \times_{\mathbb{X}_0^d} \mathbb{X}_1 \simeq \mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1$$

- Horizontal composition of arrows in **Bic**$(\mathbb{X})$ is defined by:

$$\begin{tikzcd}
A_3 \arrow[r, "f_3"] \arrow[dd, "x"'] & B_3 \arrow[r, "g_3"] \arrow[d, "y"] \arrow[dr, phantom, "\varphi_{g_3,g}"] & C_3 \arrow[d, "z"] \\
{} \arrow[r, phantom, "\varphi_{f_3,f}"] & B_2 \arrow[r, "g"'] \arrow[d, "v"] & C_2 \\
A_1 \arrow[r, "f"'] & B_1 &
\end{tikzcd}$$

Horizontal Composition of 2-Cells

$$
\begin{tikzcd}
A_6 \arrow[r, "f_6"] \arrow[ddd, "u_{62}"'] \arrow[dddr, phantom, "\varphi_{f_6,f}"] & B_6 \arrow[r, "h_6"] \arrow[d, "v_{64}"'] \arrow[dr, phantom, "\varphi_{h_6,h}"] & C_6 \arrow[d, "w_{64}"] \\
 & B_4 \arrow[r, "h"'] \arrow[d, "v_{43}"] \arrow[dr, phantom, "\beta"] & C_4 \arrow[d, "w_{43}"] \\
 & B_3 \arrow[r, "k"'] \arrow[d, "v_{32}"] \arrow[dddr, phantom, "\varphi^{-1}_{k_5,k}"] & C_3 \arrow[ddd, "w_{35}"] \\
A_2 \arrow[r, "f"] \arrow[d, "u_{21}"'] \arrow[dr, phantom, "\alpha"] & B_2 \arrow[d, "v_{21}"'] & \\
A_1 \arrow[r, "g"'] \arrow[d, "u_{15}"'] \arrow[dr, phantom, "\varphi^{-1}_{g_5,g}"] & B_1 \arrow[d, "v_{15}"] & \\
A_5 \arrow[r, "g_5"'] & B_5 \arrow[r, "k_5"'] & C_5 .
\end{tikzcd}
$$

The Double Category of Marked Paths

- Let $\mathcal{B}$ be a bicategory.
- To define the double category $\mathbf{Dbl}(\mathcal{B})$, choose a composite $A_0 \xrightarrow{\varphi_{f_1,\ldots,f_n}} A_n$ for each finite path $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n$ in $\mathcal{B}$.
- Objects of $\mathbf{Dbl}(\mathcal{B})$ are marked paths

$$A_0 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{i_0}} [A_{i_0}] \xrightarrow{f_{i_0+1}} \cdots \xrightarrow{f_n} A_n$$

- Horizontal Arrows of $\mathbf{Dbl}(\mathcal{B})$ are doubly marked paths

$$A_0 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{i_0}} [A_{i_0}] \xrightarrow{f_{i_0+1}} \cdots \xrightarrow{f_{i_1}} [A_{i_1}] \xrightarrow{f_{i_1+1}} \cdots \xrightarrow{f_n} A_n$$

The Double Category of Marked Paths

- Vertical Arrows

$$
\begin{tikzcd}
A_0 \arrow[r, "f_1"] & A_2 \arrow[r, "f_2"] & \cdots \arrow[r, "f_{i_0}"] & {[A_{i_0}]} \arrow[r, "f_{i_0+1}"] \arrow[d, equal] & \cdots \arrow[r, "f_n"] & A_n \\
B_0 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & \cdots \arrow[r, "g_{j_0}"] & {[B_{j_0}]} \arrow[r, "g_{j_0+1}"] & \cdots \arrow[r, "g_m"] & A_n
\end{tikzcd}
$$

- Double Cells

$$
\begin{tikzcd}
A_0 \arrow[r, "f_1"] & \cdots \arrow[r, "f_{i_0}"] & {[A_{i_0}]} \arrow[r, "f_{i_0+1}"] \arrow[rr, bend right, "\varphi_{f_{i_0+1},\ldots,f_{i_1}}"'] \arrow[d, equal] & \cdots \arrow[r, "f_{i_1}"] \arrow[d, phantom, "\alpha"] & {[A_{i_1}]} \arrow[r, "f_{i_1+1}"] \arrow[d, equal] & \cdots \arrow[r, "f_n"] & A_n \\
B_0 \arrow[r, "g_1"'] & \cdots \arrow[r, "g_{j_0}"'] & {[B_{j_0}]} \arrow[r, "g_{j_0+1}"'] \arrow[rr, bend left, "\varphi_{g_{j_0+1},\ldots,g_{j_1}}"] & \cdots \arrow[r, "g_{j_1}"'] & {[B_{j_1}]} \arrow[r, "g_{j_1+1}"'] & \cdots \arrow[r, "g_m"'] & B_m
\end{tikzcd}
$$

Companions

A horizontal morphisms $f\colon A \to B$ and a vertical morphism $v\colon A \mathrel{-\!\!\bullet\!\!\to} B$ are companions if there exist binding cells

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\psi"] & A \arrow[d, "\bullet" marking, "v"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\bullet" marking, "v"'] \arrow[dr, phantom, "\chi"] & B \arrow[d, equal] \\
B \arrow[r, equal] & B,
\end{tikzcd}
$$

such that

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\psi"] & A \arrow[r, "f"] \arrow[d, "\bullet" marking, "v"] \arrow[dr, phantom, "\chi"] & B \arrow[d, equal] \\
A \arrow[r, "f"'] & B \arrow[r, equal] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{id}_f"] & B \arrow[d, equal] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\psi"] & A \arrow[d, "\bullet" marking, "v"] \\
A \arrow[r, "f"'] \arrow[d, "\bullet" marking, "v"'] \arrow[dr, phantom, "\chi"] & B \arrow[d, equal] \\
B \arrow[r, equal] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "\bullet" marking, "v"'] \arrow[dr, phantom, "1_v"] & A \arrow[d, "\bullet" marking, "v"] \\
B \arrow[r, equal] & B
\end{tikzcd}
.
$$

What kind of arrow in a bicategory would correspond to a companion in a weakly globular double category?

Companions and Quasi Units

Definition

An arrow $f\colon A \to A$ in a bicategory $\mathcal{B}$ is a quasi unit if $f \cong 1_A$.

Proposition

A horizontal arrow $w\colon A \to B$ has a companion if and only if $w\colon \bar{A} \to \bar{B}$ is a quasi unit in $\mathbf{Bic}\mathbb{X}$.

Proposition

A horizontal arrow

$$A_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i_0}} [A_{i_0}] \xrightarrow{f_{i_0+1}} \cdots \xrightarrow{f_{i_1}} [A_{i_1}] \xrightarrow{f_{i_1}+1} \cdots \xrightarrow{f_n} A_n$$

in $\mathbf{Dbl}(\mathcal{B})$ *has a companion if and only if* $A_{i_0} = A_{i_1}$ *and the chosen composition* $\varphi_{f_{i_0+1}\cdots f_{i_1}}$ *is a quasi unit.*

Proposition

Let $F \colon \mathbb{X} \to \mathbb{Y}$ *be a pseudo-functor of weakly globular double categories. If a horizontal arrow* f *in* $\mathbb{X}$ *has a companion then so does* $F(f)$ *in* $\mathbb{Y}$*.*

Internal equivalences

What type of horizontal arrow in a weakly globular double category corresponds to the internal equivalences in a bicategory?

Pre-companions

A horizontal arrow $A \xrightarrow{f} B$ is a left pre-companion if there are $A' \xrightarrow{f'} B'$ and $B' \xrightarrow{r_f} C$ with a vertically invertible double cell

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\bullet" marking] \arrow[dr, phantom, "\varphi"] & B \arrow[d, "\bullet" marking] & \\
A' \arrow[r, "f'"'] & B' \arrow[r, "r_f"'] & C
\end{tikzcd}$$

such that $r_f \circ f'$ is a companion.

Pre-companions

A horizontal arrow $A \xrightarrow{f} B$ is a right pre-companion if there are $A'' \xrightarrow{f''} B''$ and $D \xrightarrow{l_f} A''$ with a vertically invertible double cell

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "\bullet" marking] \arrow[dr, phantom, "\varphi'"] & B \arrow[d, "\bullet" marking] \\
D \arrow[r, "l_f"'] & A'' \arrow[r, "f''"'] & B''
\end{tikzcd}$$

such that $f'' \circ l_f$ is a companion in $\mathbb{X}$.

A horizontal arrow $A \xrightarrow{f} B$ is a pre-companion if it is both a left and a right pre-companion.

Pre-companions and Equivalences

Proposition

A horizontal arrow

$$A_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i_0}} [A_{i_0}] \xrightarrow{f_{i_0+1}} \cdots \xrightarrow{f_{i_1}} [A_{i_1}] \xrightarrow{f_{i_1}+1} \cdots \xrightarrow{f_n} A_n$$

in $\mathbf{Dbl}(\mathcal{B})$ *is a pre-companion if and only if the chosen composition* $\varphi_{f_{i_0+1}\cdots f_{i_1}}$ *is an equivalence in* $\mathcal{B}$*.*

Proposition

A horizontal arrow $f\colon A \to B$ *in* $\mathbb{X}$ *is a pre-companion if and only if the arrow* $f\colon \bar{A} \to \bar{B}$ *in* $\mathbf{Bic}(\mathbb{X})$ *is an equivalence.*

Categories of fractions

Let $\mathbf{C}$ be a category with a class W of arrows admitting a calculus of fractions. Now we can form:

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "I_{\mathbf{C}}"] \arrow[dr, "J_{\mathbf{C}}"'] & \mathbf{C}[W^{-1}] \\
 & \mathbf{C}(W^{-1}) \arrow[u, two heads]
\end{tikzcd}$$

the category of fractions

the bicategory of fractions

$$\begin{tikzcd}
\mathbf{Dbl}(\mathbf{C}) \arrow[r, "\mathbf{Dbl}(J_{\mathbf{C}})"] & \mathbf{Dbl}(\mathbf{C}(W^{-1})) \\
H\mathbf{C} \arrow[u, "\simeq_2"'] \arrow[ur] &
\end{tikzcd}$$

weakly globular double

category of fractions?

The Vertical Universal Property of $\mathbf{Dbl}(\mathbf{C}(W^{-1}))$

Theorem

Composition with

$$\mathbf{Dbl}(\mathbf{C}) \xrightarrow{\mathbf{Dbl}(J_{\mathbf{C}})} \mathbf{Dbl}(\mathbf{C}(W^{-1}))$$

gives a biequivalence of 2-categories

$$\mathbf{WGDbl}_{\mathrm{ps},W}(\mathbf{Dbl}(\mathbf{C}), \mathbb{D}) \simeq \mathbf{WGDbl}_{\mathrm{ps}}(\mathbf{Dbl}(\mathbf{C}(W^{-1})), \mathbb{D})$$

where the objects of $\mathbf{WGDbl}_{\mathrm{ps},W}(\mathbf{Dbl}(\mathbf{C}), \mathbb{D})$ *are pseudo functors which send horizontal arrows corresponding to elements of* W *to pre-companions.*

Remarks

- Since $H\mathbf{C} \xrightarrow{\simeq_2} \mathbf{Dbl}(\mathbf{C})$, this translates to

$$\mathbf{WGDbl}_{\mathrm{ps},W}(H\mathbf{C},\mathbb{D}) \simeq \mathbf{WGDbl}_{\mathrm{ps}}(\mathbf{Dbl}(\mathbf{C}(W^{-1})),\mathbb{D})$$

- However, neither $\mathbf{Dbl}(J_{\mathbf{C}})$ nor $H\mathbf{C} \longrightarrow \mathbf{Dbl}(\mathbf{C}(W^{-1}))$ are strict functors, so there is no horizontal universal property for these arrows.
- We want to find $\mathbf{C}\{W\} \simeq_2 \mathbf{Dbl}(\mathbf{C}(W^{-1}))$ (vertically equivalent) such that $H\mathbf{C} \to \mathbf{C}\{W\}$ is a strict functor.

$\mathbf{C}\{W\}$

- Objects are arrows in W, $(w) = \left(A \xrightarrow{w} B\right)$.
- A vertical arrow $(u_1, C, u_2) \colon (A_1 \xrightarrow{w_1} B) \longrightarrow\!\!\bullet\!\!\to (A_2 \xrightarrow{w_2} B)$ is an equivalence class of commutative diagrams

$$\begin{tikzcd} A_1 \arrow[r, "w_1"] & B \arrow[dd, equal] \\ C \arrow[u, "u_1"] \arrow[d, "u_2"'] & \\ A_2 \arrow[r, "w_2"'] & B \end{tikzcd}$$

$(u_1, C, u_2) \sim (v_1, D, v_2)$ if

$$\begin{tikzcd} & A_1 & \\ C \arrow[ur, "u_1"] \arrow[dr, "u_2"'] & E \arrow[l, "r_1"'] \arrow[r, "r_2"] & D \arrow[ul, "v_1"'] \arrow[dl, "v_2"] \\ & A_2 & \end{tikzcd}$$

$\mathbf{C}\{W\}$

A horizontal arrow

$$(A \xrightarrow{w} B) \xrightarrow{f} (A' \xrightarrow{w'} B')$$

is given by an arrow $A \xrightarrow{f} A'$ in **C**. We draw this as

$$(B \xleftarrow{w} A) \xrightarrow{f} (A' \xrightarrow{w'} B')$$

.

$\mathbf{C}\{W\}$

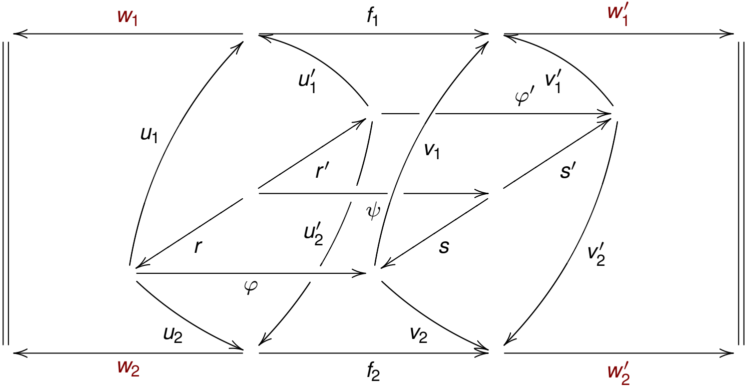
A double cell

$$\begin{tikzcd}
(w_1) \arrow[r, "f_1"] \arrow[d, "{[u_1, C, u_2]}"' , "\bullet" marking] \arrow[dr, phantom, "(\varphi)"] & (w'_1) \arrow[d, "{[u'_1, C', u'_2]}", "\bullet" marking] \\
(w_2) \arrow[r, "f_2"'] & (w'_2)
\end{tikzcd}$$

is an equivalence class of

$$\begin{tikzcd}
B & A_1 \arrow[l, "w_1"'] \arrow[r, "f_1"] & A'_1 \arrow[r, "w'_1"] & B' \\
& C \arrow[u, "u_1"'] \arrow[d, "u_2"] \arrow[r, "\varphi_{u_1, u_2, u'_1, u'_2}"'] & C' \arrow[u, "u'_1"] \arrow[d, "u'_2"] & \\
B \arrow[uu, equal] & A_2 \arrow[l, "w_2"] \arrow[r, "f_2"'] & A'_2 \arrow[r, "w'_2"'] & B' \arrow[uu, equal]
\end{tikzcd}$$

Equivalence relation on double cells



Properties of this equivalence relation

A double cell may not have a representative for each combination of representatives of its vertical domain and codomain arrows. However, given a double cell

$$
\begin{tikzcd}
B \arrow[dd, equal] & A_1 \arrow[l, "w_1"'] \arrow[r, "f_1"] & A'_1 \arrow[r, "w'_1"] & B' \arrow[dd, equal] \\
 & C \arrow[u, "u_1"'] \arrow[d, "u_2"'] \arrow[r, "\varphi_{u_1, u_2, u'_1, u'_2}"'] & C' \arrow[u, "u'_1"'] \arrow[d, "u'_2"'] & \\
B & A_2 \arrow[l, "w_2"'] \arrow[r, "f_2"'] & A'_2 \arrow[r, "w'_2"'] & B'
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
 & A_1 & \\
C \arrow[ur, "u'_1"] \arrow[dr, "u'_2"'] & E \arrow[l, "r_1"'] \arrow[r, "r_2"] & D \arrow[ul, "v_1"'] \arrow[dl, "v_2"] \\
 & A_2 &
\end{tikzcd}
$$

then there is an arrow $\tilde{r}$ with a representative of the double cell of the form

$$
\begin{tikzcd}
B \arrow[dd, equal] & A_1 \arrow[l, "w_1"'] \arrow[r, "f_1"] & A_1' \arrow[r, "w_1'"] & B' \arrow[dd, equal] \\
 & C \arrow[u, "u_1\tilde{r}"'] \arrow[d, "u_2\tilde{r}"] \arrow[r, "\varphi_{u_1\tilde{r}, u_2\tilde{r}, u_1', u_2'}"'] & C' \arrow[u, "v_1"'] \arrow[d, "v_2"] & \\
B & A_2 \arrow[l, "w_2"] \arrow[r, "f_2"'] & A_2' \arrow[r, "w_2'"'] & B'
\end{tikzcd}
$$

Remarks

- There is a vertical arrow
  $(A \xrightarrow{w} B) \longrightarrow\!\!\!\!\bullet\!\!\!\!\!\!\rightarrow (A' \xrightarrow{w'} B')$ if and only if $B = B'$
  and in that case there is precisely one vertical arrow.
- There is at most one double cell for any square 'frame' of horizontal and vertical arrows.
- All double cells in $\mathbf{C}\{W\}$ are vertically invertible.
- The codomain functor $d_1 : \mathbf{C}\{W\}_1 \to \mathbf{C}\{W\}_0$ is an isofibration.

$\mathbf{C}(W^{-1})$ and $\mathbf{C}\{W\}$

Theorem

There is a 2-equivalence of bicategories

$$\mathbf{Bic}(\mathbf{C}\{W\}) \simeq \mathbf{C}(W^{-1})$$

and there is a vertical 2-equivalence of weakly globular double categories

$$\mathbf{C}\{W\} \simeq_{2,v} \mathbf{Dbl}(\mathbf{C}(W^{-1})).$$

$\mathbf{C}(W^{-1})$ and $\mathbf{C}\{W\}$

Remark

There is a strict functor $\mathcal{J}_{\mathbf{C}} \colon H\mathbf{C} \to \mathbf{C}\{W\}$, mapping A to (1_A) and $A \xrightarrow{f} B$ to $(1_A) \xrightarrow{f} (1_B)$.

Corollary

(The Vertical Universal Property) Composition with $\mathcal{J}_{\mathbf{C}}$ induces an equivalence of categories,

$$\mathbf{WGDbl}_{\mathrm{ps},W}(H\mathbf{C}, \mathbb{D}) \simeq \mathbf{WGDbl}_{\mathrm{ps}}(\mathbf{C}\{W\}, \mathbb{D})$$

What is the horizontal universal property of $\mathbf{C}\{W\}$?

Companions in $\mathbf{C}\{W\}$

$$\psi_{u,w} = \begin{tikzcd} (B \arrow[dd, equal] & A_1) \arrow[l, "wu"'] \arrow[r, equal] \arrow[d, equal] & (A_1 \arrow[r, "wu"] \arrow[d, equal] & B) \arrow[dd, equal] \\ & A_1 \arrow[r, equal] \arrow[d, equal] & A_1 \arrow[d, "u"'] & \\ (B & A_1) \arrow[l, "wu"] \arrow[r, "u"'] & (A_2 \arrow[r, "w"'] & B) \end{tikzcd}$$

$$\chi_{u,w} = \begin{tikzcd} (B \arrow[dd, equal] & A_1) \arrow[l, "wu"'] \arrow[r, "u"] \arrow[d, equal] & (A_2 \arrow[r, "w"] \arrow[d, equal] & B) \arrow[dd, equal] \\ & A_1 \arrow[r, "u"] \arrow[d, "u"'] & A_2 \arrow[d, equal] & \\ (B & A_2) \arrow[l, "w"] \arrow[r, equal] & (A_2 \arrow[r, "w"'] & B) \end{tikzcd}$$

Companions in $\mathbf{C}\{W\}$

- The vertical companion arrows and their inverses generate the vertical arrow category.
- The binding cells of the companion pairs together with their vertical inverses generate all the double cells in $\mathbf{C}\{W\}$.

The Horizontal Universal Property

Theorem

Composition with $\mathcal{J}_{\mathbf{C}}\colon H\mathbf{C} \to \mathbf{C}\{W\}$ induces an equivalence of categories

$$\mathbf{WGDbl}_h(\mathbf{C}\{W\}, \mathbb{D}) \simeq \mathbf{WGDbl}_{h,W}(H\mathbf{C}, \mathbb{D}),$$

where $\mathbf{WGDbl}_{h,W}(H\mathbf{C}, \mathbb{D})$ is the category of W-friendly functors and W-friendly horizontal transformations.