Two-dimensional Morita theory and Galois cohomology

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Outline

This is work in progress, joint with Evan Jenkins

Groups associated to a commutative ring

Throughout, R will denote a commutative ring

Groups constructed from rings

- $R^\times$, the group of units
- $\mathrm{Pic}(R)$, the Picard group: invertible R-modules up to isomorphism

$$M \otimes N \cong R \quad \text{and} \quad R \cong N \otimes M$$

- $\mathrm{Br}(R)$, the Brauer group: Azumaya algebras up to Morita equivalence

$$A \otimes B \simeq R \quad \text{and} \quad R \simeq B \otimes A$$

Example: $\mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2$, generated by $\mathbb{H}$

Classical Morita theory

The bicategory Prof_1 of profunctors

0-cells: R-algebras
Hom-categories: $\mathrm{Prof}_1(A, B)$ is the category of A-B-bimodules
Composition: tensor product of bimodules

Equivalence in the bicategory of profunctors is called *Morita equivalence*.

Fact

A and B are Morita equivalent if and only if Mod_A and Mod_B are equivalent as R-linear categories.

Tensor product over R turns Prof_1 into a symmetric monoidal bicategory.

Relating $R^\times$, $\mathrm{Pic}(R)$, and $\mathrm{Br}(R)$

Definition

The *core* of a monoidal bicategory $\mathscr{B}$ is the monoidal bicategory with:
0-cells: invertible objects in $\mathscr{B}$
1-cells: equivalences in $\mathscr{B}$
2-cells: invertible 2-cells in $\mathscr{B}$
The core of $\mathscr{B}$ is denoted by $\mathrm{Core}(\mathscr{B})$.

Observation (Street)

Let X denote the nerve of the 1-object tricategory $\mathrm{Core}(\mathrm{Prof}_1)$, with basepoint given by the unit object. Then there are equivalences

$$\mathrm{Br}(R) \cong \pi_0(X) \qquad \mathrm{Pic}(R) \cong \pi_1(X) \qquad R^\times \cong \pi_2(X)$$

and all other homotopy groups of X are trivial.

Why stop with the Brauer group?

Fact

For $R = K$ a field, these three groups coincide with the first three Galois cohomology groups:

$$K^\times = H^0(\mathrm{Gal}(\overline{K}/K), \overline{K}^\times) \qquad \mathrm{Pic}(K) = H^1 \qquad \mathrm{Br}(K) = H^2$$

Question

Can we find a higher category associated to K which allows us to extend Street's result to the third Galois cohomology group?

Idea: try to categorify!

R-algebra $\rightsquigarrow$ R-linear monoidal category

Azumaya algebra $\rightsquigarrow$ 2-Morita invertible R-linear monoidal category

Finding the right setting

From work of Garner and Shulman, we (almost) get a theory of 2-profunctors for pseudomonoids in a sufficiently cocomplete monoidal bicategory $\mathscr{B}$.

Goal

Starting with K, we want to construct a monoidal bicategory $\mathscr{B}$ such that the nerve X of $\mathrm{Core}\big(\mathrm{Prof}_2(\mathscr{B})\big)$ satisfies:

$$\pi_0(X) \cong H^3\big(\mathrm{Gal}(\overline{K}/K), \overline{K}^\times\big)$$

$$\pi_1(X) \cong H^2\big(\mathrm{Gal}(\overline{K}/K), \overline{K}^\times\big) \cong \mathrm{Br}(K)$$

$$\pi_2(X) \cong H^1\big(\mathrm{Gal}(\overline{K}/K), \overline{K}^\times\big) \cong \mathrm{Pic}(K) = 0$$

$$\pi_3(X) \cong H^0\big(\mathrm{Gal}(\overline{K}/K), \overline{K}^\times\big) \cong K^\times$$

and $\pi_n(X) = 0$ for $n > 3$.

Finding the right setting

First idea: Take $\mathscr{B}$ to be the monoidal bicategory of R-linear categories.

Problem

For a monoidal category $\mathscr{V}$, we have $\mathrm{Prof}_1(\mathscr{V})(I,I) \simeq \mathscr{V}$. Therefore we expect that $\mathrm{Prof}_2(\mathscr{B})(I,I) \simeq \mathscr{B}$. The nerve of the endomorphism category is equivalent to the loop space:

$$N\mathscr{C}(c,c) \simeq \Omega N\mathscr{C}$$

Therefore π_1 of our space X is the group of invertible objects in $\mathscr{B}$.

If we want $\pi_1(X) = \mathrm{Br}(K)$, we need a monoidal bicategory $\mathscr{B}$ such that:

- equivalence in $\mathscr{B}$ is ordinary Morita equivalence
- $\mathscr{B}$ is nice enough to admit the construction of two-dimensional profunctors

Finding the right setting

Solution: Take $\mathscr{B}$ to be the symmetric monoidal bicategory $\mathbf{Cat}_R^{cc}$ of *Cauchy complete* R-linear categories.

Theorem (Jenkins, S)

The monoidal bicategories $\mathrm{Core}(\mathbf{Cat}_K^{cc})$ and $\mathrm{Core}(\mathrm{Prof}_1)$ are equivalent.

Proof idea: The equivalence sends A to its Cauchy completion Mod_A^{fgp}
Difficulty: showing that every invertible Cauchy complete K-linear category is of this form

Consequence

At least for $R = K$ a field, the loop space ΩX of our space X is equivalent to the space $N\mathrm{Core}(\mathrm{Prof}_1)$ considered by Street.

Constructing a monoidal category from a cocycle

Fix a finite Galois extension E/K, with Galois group G, and a (normalized) 3-cocycle

$$\omega\colon G \times G \times G \to E^{\times}$$

Construction

Let

$$\mathscr{A} = \prod_{g \in G} \mathrm{Vec}_E$$

generated by simple objects S_g indexed by $g \in G$. Tensor product on objects: $S_g \otimes S_h = S_{gh}$.
All morphisms are matrices with entries in $\mathscr{A}(S_g, S_g) = E$. Tensor product of morphisms: if $\lambda\colon S_g \to S_g$, $\mu\colon S_h \to S_h$, then $\lambda \otimes \mu = \lambda g(\mu)$.
Associator given by $\omega(g,h,k)\colon S_{ghk} \to S_{ghk}$

$\mathscr{A}$ is an Azumaya 2-algebra

Definition

An Azumaya 2-algebra is an invertible object in $\mathrm{Prof}_2(\mathbf{Cat}_R^{cc})$.

Fact

If A is a projective generator of $\mathrm{Mod}_{A \otimes A^{\mathrm{op}}}$ and $Z(A) = R$, then $A \otimes A^{\mathrm{op}} \simeq R$ and A is an Azumaya algebra

The proof of this is very robust, so it categorifies. Therefore we need to show that:

- The center of $\mathscr{A}$ is equivalent to Vec_K
- $\mathscr{A}$ is a projective generator of the 2-category of $\mathscr{A}$-$\mathscr{A}$-bimodules

The center of $\mathscr{A}$

Claim

The center of $\mathscr{A}$ is equivalent to Vec_K

Proof: An application of Hilbert's Theorem 90.

Projectivity of $\mathscr{A}$

For any algebra A, the multiplication $A \otimes A^{\mathrm{op}}$ is surjective and it is a morphism of $A \otimes A^{\mathrm{op}}$-modules. Therefore A is projective if and only if the multiplication is split. Such algebras are called *separable*.

Problem

The multiplication $\mathscr{A} \otimes \mathscr{A}^{\mathrm{rev}} \to \mathscr{A}$ of $\mathscr{A}$ is not split.

Reason

Note: $\mathscr{P}$ finitely generated projective *must* mean that

$$\mathrm{Hom}_{\mathscr{A} \otimes \mathscr{A}^{\mathrm{rev}}}(\mathscr{P}, -)$$

preserves colimits. This does *not* imply that it preserves essentially surjective functors.

Projectivity of $\mathscr{A}$

Solution

Show that $\mathscr{A} \otimes \mathscr{A}^{\mathrm{rev}} \to \mathscr{A}$ exhibits $\mathscr{A}$ as (absolute) Kleisli object.

Proposition (Jenkins, S)

The assignment which sends a Galois cocycle ω to

$$\mathscr{A} = \prod_{g \in G} \mathrm{Vec}_E$$

gives a well-defined map

$$H^3\big(\mathrm{Gal}(\overline{K}/K), \overline{K}^{\times}\big) \to \pi_0 X$$

Why stop at $n = 3$?

Let $\mathscr{V}_1$ be symmetric monoidal. Using a hypothetic version of higher profunctors, we could form

$$
\begin{array}{ccccccc}
 & & & \mathscr{V}_1 & & & \\
 & \swarrow & & & \searrow & & \\
\mathscr{V}_1 - \mathbf{Cat} & \longrightarrow & \mathscr{V}_1 - \mathbf{Cat}^{cc} =: \mathscr{V}_2 & & & \mathrm{Prof}_1 & \longrightarrow \mathrm{Core}_1 \\
 & \swarrow & & \searrow & & & \\
\mathscr{V}_2 - \mathbf{Cat} & \longrightarrow & \mathscr{V}_2 - \mathbf{Cat}^{cc} =: \mathscr{V}_3 & & & \mathrm{Prof}_2 & \longrightarrow \mathrm{Core}_2 \\
 & \swarrow & & \searrow & & & \\
\ldots & \longrightarrow & \ldots & & & \mathrm{Prof}_3 & \longrightarrow \mathrm{Core}_3 \\
 & & & \searrow & & & \\
 & & & & & \ldots & \longrightarrow \ldots
\end{array}
$$

Why stop at $n = 3$?

Question

Do we get an equivalence $\Omega N\mathrm{Core}_{n+1} \simeq N\mathrm{Core}_n$?

Question

If $\mathscr{V} = \mathrm{Vect}_K$, do we get isomorphisms

$$\pi_0 \mathrm{Core}_n \cong H^{n+1}\big(\mathrm{Gal}(\overline{K}/K), \overline{K}^\times\big)?$$

Question

If $\mathscr{V} = \mathrm{Mod}_R$, do we get isomorphisms

$$\pi_0 \mathrm{Core}_n \cong H^{n+1}_{\mathrm{et}}\big(\mathrm{Spec}(R), \mathbb{G}_m\big)?$$

Thanks!