Weak n-categories with strict units via iterated enrichment

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joint with
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n-operads

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An n-operad is a cartesian monad morphism $A \to \mathcal{T}_{\leq n}$. The fibre $A_p \subseteq A_k$ over $p \in (\mathcal{T}_{\leq n}1)_k$ is the set of ways of composing p to a k-cell in any A-algebra.

n-operads

For $p = \bullet \to \bullet \to \bullet \to \bullet \in (\mathcal{T}_{\leq 2})_1$ and A the operad for bicategories in the sense of Bénabou, A_p includes 2 elements corresponding to the composites $(f \circ g) \circ h$ and $f \circ (g \circ h)$.

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In the above example, A_{zp} is the set of coherence 2-cells between alternative ways of composing p in any bicategory.

n-operads over **Set**

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For such an A and $n > 1$,

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defines a distributive lax monoidal structure on $\mathcal{G}^{n-1}\mathbf{Set}$, whose associated enriched categories are exactly A-algebras.

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The case $m = 1$ gives a monad A_1 on $\mathcal{G}^{n-1}\mathbf{Set}$, which encodes the structure possessed by the homs of an A-algebra.

Lifting theorem

Theorem

Let E be an accessible distributive lax monoidal structure on a cocomplete category V. Then there is, up to isomorphism, a unique lax monoidal structure E' on E_1-Alg such that

1. E'_1 *is the identity.*
2. E' *is distributive.*
3. E'-Cat $\cong$ E-Cat *over* $\mathcal{G}V$.

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Question: When can we iterate this by applying the same constructions to A_1 in place of A?

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For $\mathcal{B}$ a bicategory and $x, y \in \mathcal{B}$, $\mathcal{B}(x, y)$ is a category. But it has more structure than that.

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Obstruction: The presence of weak units causes the 1-operad which describes the structure enjoyed by the homs of a bicategory **NOT** to be over **Set**.

Reduced n-operads

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Reduced operads are over **Set** and for $n > 0$, if A is reduced then so is A_1. Thus if A is a reduced operad, then A-algebras are definable by successive enrichment.

Reduced n-collections

For a general n-operad A the sets A_p form a presheaf on the category $\mathrm{el}(\mathcal{T}_{\leq n}1)$, which is the **underlying n-collection** of A.

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For reduced A, we define its underlying **reduced** n-collection to be the associated presheaf on the full subcategory $\mathcal{R}_{\leq n}$ of $\mathrm{el}(\mathcal{T}_{\leq n}1)$ consisting of those p **not** of the form $z^r U_k$.

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Let $\mathrm{RGr}_{\leq n}$ be the n-operad for reflexive n-globular sets. It is subterminal. An n-operad (or n-collection) A is reduced iff the projection $A \times \mathrm{RGr}_{\leq n} \to \mathrm{RGr}_{\leq n}$ is an isomorphism.

Theory of reduced n-operads

We have

$$\begin{tikzcd}
\text{Rd-}n\text{-Op} \arrow[r, "J"] \arrow[d, "U^{(\mathrm{rd})}"'] \arrow[dr, phantom, "\mathrm{pb}"] & n\text{-Op} \arrow[d, "U"] \\
\text{Rd-}n\text{-Coll} \arrow[r, "I"'] & n\text{-Coll}
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times \mathrm{RGr}_{\leq n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{po}"] & A \arrow[d, "\pi_A"] \\
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where $F \dashv U$ and $L \dashv I$. Thus

1. Rd-n-Op is finitarily monadic over Rd-n-Coll.
2. The left adjoint to J is obtained by taking the colimit of the sequence $1 \xrightarrow{\pi} P \xrightarrow{P\pi} P^2 \to \ldots$ by an appropriate application of Wolff's theorem.

Theory of reduced n-operads

For $n > 0$ we have functors

$$h : n\text{-}\mathrm{Op}_{/\mathbf{Set}} \longrightarrow (n{-}1)\text{-}\mathrm{Op} \qquad r : (n{-}1)\text{-}\mathrm{Op} \longrightarrow n\text{-}\mathrm{Op}_{/\mathbf{Set}}$$

where

- hA-algebra structure = the structure that the homs of a A-algebra has.
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For all $(n{-}1)$-operads B, $rhB = B$.

Theory of reduced n-operads

A pasting diagram p of dimension $k > 0$ can be described in terms of pasting diagrams of dimension $k-1$

$$p = (p_1, ..., p_m) = 0 \xrightarrow{p_1} ... \xrightarrow{p_m} m$$

and in these terms

$$hr(A)_p = \prod_{i=1}^{m} A_{(p_i)}.$$

Lifted tensor product $\to$ cartesian product

When A is reduced one can substitute the unique unit operations of A in an appropriate way to produce functions $A_p \to A_{(p_i)}$, and thus an operad map $\nu_A : A \to hrA$.

Proposition

1. *h and r restrict to an adjunction $h \dashv r$ between categories of reduced operads with unit ν.*
2. *ν_A induces a canonical comparison $A' \to \prod$.*

Pointed reduced collections

The underlying **pointed reduced n-collection** of a reduced n-operad A includes its underlying reduced collection, but also remembers the functions $A_p \to A_{(p_i)}$.

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A morphism $f : p \to q$ of k-stage trees is a commutative diagrams of sets

$$\begin{tikzcd} p^{(k)} \arrow[r, "\partial_k"] \arrow[d, "f^{(k)}"'] & p^{(k-1)} \arrow[r] \arrow[d, "f^{(k-1)}"'] & \cdots \arrow[r] & p^{(1)} \arrow[r, "\partial_1"] \arrow[d, "f^{(1)}"] & p^{(0)} \arrow[d, "f^{(0)}"] \\ q^{(k)} \arrow[r, "\partial_k"'] & q^{(k-1)} \arrow[r] & \cdots \arrow[r] & q^{(1)} \arrow[r, "\partial_1"'] & q^{(0)} \end{tikzcd}$$

such that for $0 < i \leq k$, $f^{(i)}$ is order preserving on the fibres of ∂_i.

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- $\Psi_{\leq n} \subseteq \mathrm{el}(\Omega^{(\mathrm{lin})}_{\leq n})$ is the full subcategory consisting of pasting diagrams not of the form $z^r U_k$. A pointed reduced collection is a presheaf on $\Psi_{\leq n}$.

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- $\Psi_{\leq n} \subseteq \text{el}(\Omega^{(\text{lin})}_{\leq n})$ is the full subcategory consisting of pasting diagrams not of the form $z^r U_k$. A pointed reduced collection is a presheaf on $\Psi_{\leq n}$.
- $\Psi_{\leq n}$ has two special kinds of morphisms. Those of $\mathcal{R}_{\leq n}$ which are cosource and cotarget maps from $\text{el}(\Omega_{\leq n})$, and those corresponding to tree inclusions whose corresponding subcategory we call $\mathcal{V}_{\leq n}$.

Contractibility in terms of awfs's

Garner's description of contractibility uses an awfs cofibrantly generated by

$$\mathrm{obj}(\mathrm{el}(\mathcal{T}_{\leq n}1)) \to (n\text{-}\mathrm{Coll}_{/\mathbf{Set}})^{\to} \qquad p \mapsto \partial(p) \hookrightarrow p$$

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A contractible n-collection over **Set** is an algebra of the resulting fibrant replacement monad, the data of which involves giving a choice of composition operations and coherence operations.

Reduced-contractibility

Similarly contractibility for pointed reduced collections is defined using an awfs cofibrantly generated by

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The effect of the arrows in $\mathcal{V}_{\leq n}$ is that the choices of operations one has for reduced-contractibility must be compatible with the process of substituting in unit operations.

Low dimensional examples

Example

The 2-operad for Bénabou bicategories with strict units is reduced-contractible in two ways, with chosen operations obtained by bracketting completely to the left, or to the right.

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Example

The 3-operad for Gray categories may also be exhibited as reduced-contractible in two ways.

Main Theorem

Theorem

1. *The initial reduced-contractible n-operad* $\mathcal{K}^{(su)}_{\leq n}$ *exists.*
2. *For* $n > 0$, $h(\mathcal{K}^{(su)}_{\leq n}) = \mathcal{K}^{(su)}_{\leq n-1}$.

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2. *For* $n > 0$, $h(\mathcal{K}^{(su)}_{\leq n}) = \mathcal{K}^{(su)}_{\leq n-1}$.

Thus there is a (lax) tensor product of weak $(n-1)$-categories with strict units, whose enriched categories are exactly weak n-categories with strict units.

Proof of Main Theorem

Indication of proof:

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(2): The adjunction $h \dashv r$ works at the level of pointed reduced collections, and then lifts to an adjunction between categories of reduced-contractible operads. So this lifted h preserves initial objects since it is a left adjoint.