Betti numbers via linear algebra

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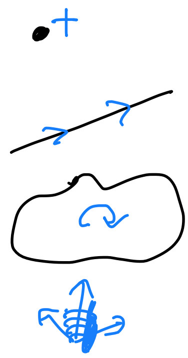
Given a manifold X decomposed into finitely many cells (a "CW-complex").

c_n = # of n-dim. cells, labeled $e_1^n, \dots, e_{c_n}^n$

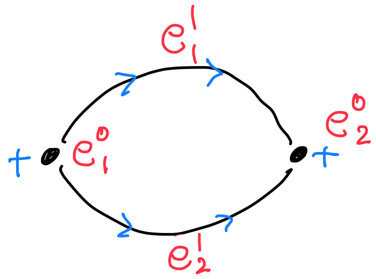
Each cell is given with an orientation.
These need not "match up" in any way.

Recall:

n	orientation
0	
1	
2	
3	

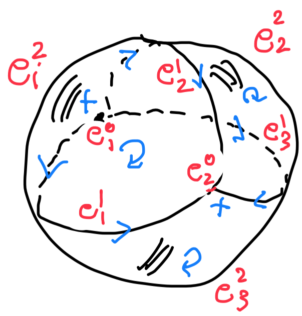


e.g.s S^1



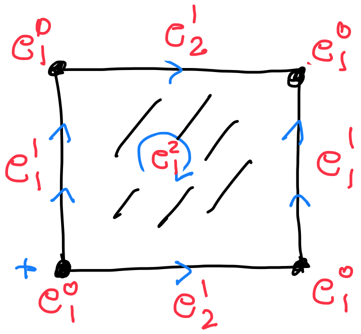
$c_0 = 2$
$c_1 = 2$

S^2



$c_0 = 2$
$c_1 = 3$
$c_2 = 3$

T^2



$c_0 = 1$
$c_1 = 2$
$c_2 = 1$

When is there an n-dim "hole" in X?

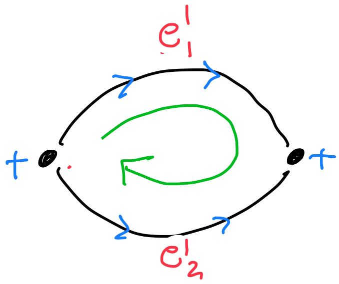
When there's an n-dim "closed manifold" inside X that isn't the boundary of some $(n+1)$-dim manifold-with-boundary inside X.

Both are put together from cells of X.

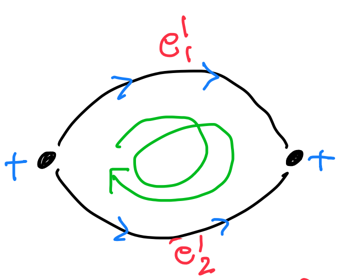
If you worry about <u>how</u> they're put together, you get <u>homotopy groups</u>. But it's easier to just ignore it!

An <u>n-chain</u> is a C_n-dimensional vector. The i^{th} component counts how many times the i^{th} n-cell appears, w/ opposite orientations negative.

e.g.s

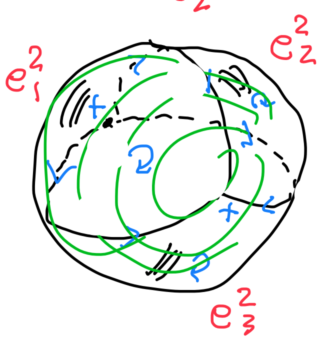


traverses e^1_1 forwards & e^1_2 backwards $\rightsquigarrow$ $(1,-1)$



$(2,-2)$

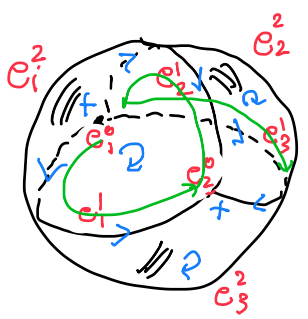
<u>NB</u> We allow non-integral components "formally" so we can use linear algebra instead of abstract algebra!



$(1,1,1)$

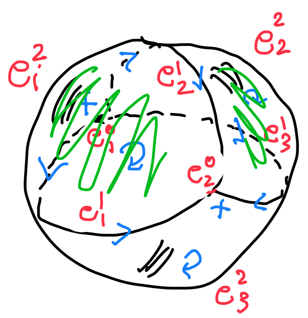
Chains can have a boundary

eg



A 1-chain

$(1,-1,1)$

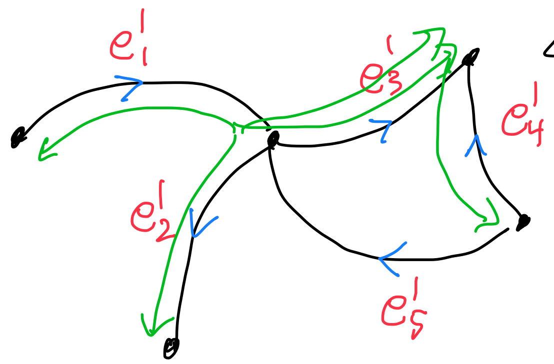


A 2-chain

$(1,1,0)$

Chains can be weird and non-manifoldy.

eg In X:



1-chains

$(-1,1,2,-1,0)$

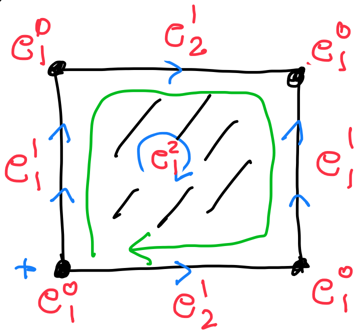
or even worse:

$(2,0,\frac{3}{2},-1,\pi)$

(not drawable!)

Chains can lose geometric information

eg

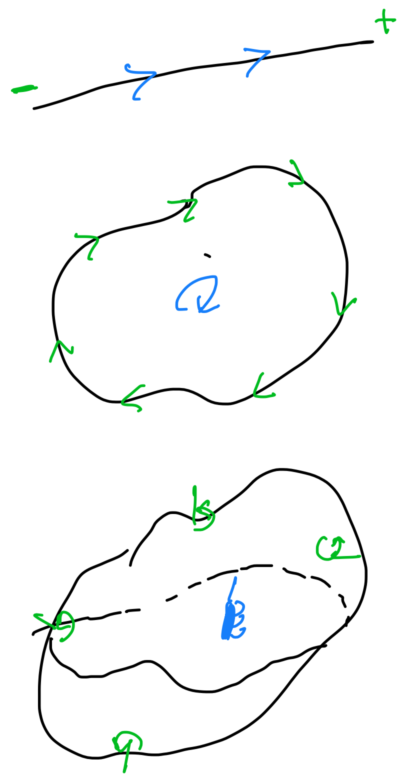


$$+e_1^1+e_2^1-e_1^1-e_2^1$$

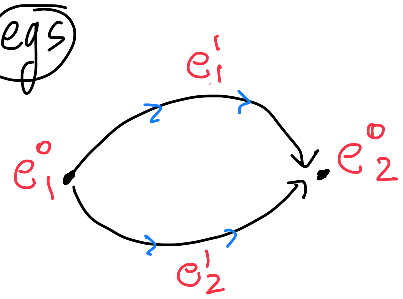
$$=(0,0) \quad !$$

The boundary of each n-cell is an $(n-1)$-chain!

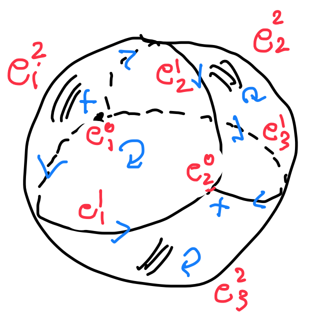
Recall "boundary orientation"



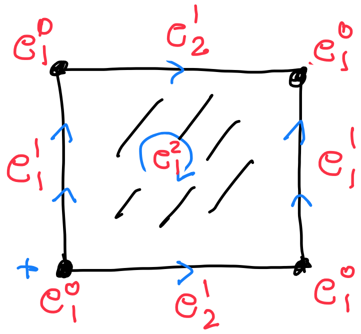
egs



$$\partial e_1^1 = (-1, 1)$$
$$\partial e_2^1 = (-1, 1)$$



$$\partial e_1^2 = (-1, 1, 0)$$
$$\partial e_2^2 = (0, -1, 1)$$
$$\partial e_3^2 = (1, 0, -1)$$
$$\partial e_1^1 = \partial e_2^1 = \partial e_3^1 = (-1, 1)$$

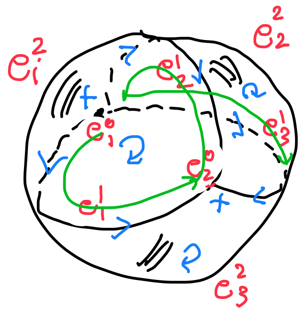


$$\partial e_1^2 = (0, 0)$$
$$\partial e_1^1 = \partial e_2^1 = (0).$$

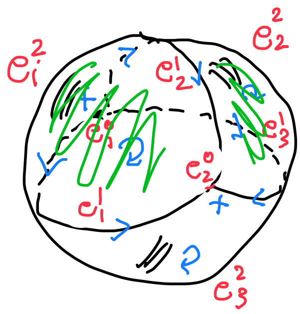
Therefore, the boundary of an n-chain is also an $(n-1)$-chain.

$$\partial^n(a_1, \dots, a_{c_n}) = a_1 \cdot \partial e_1^n + a_2 \cdot \partial e_2^n + \dots + a_{c_n} \cdot \partial e_{c_n}^n .$$

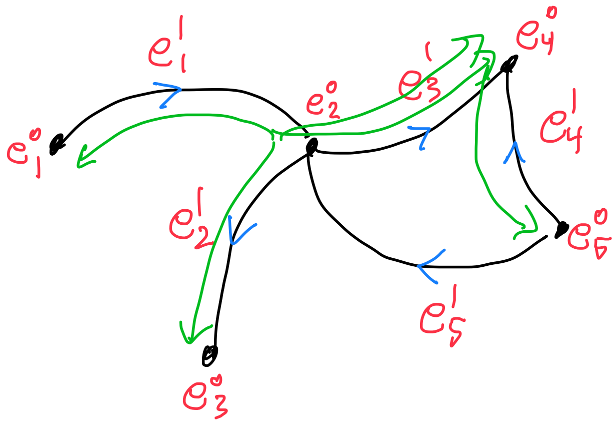
egs



$$\partial^1(1,-1,1) = 1\cdot(-1,1) - 1\cdot(-1,1) + 1\cdot(-1,1)$$
$$= (-1,1)$$



$$\partial^2(1,1,0) = 1\cdot(-1,1,0) + 1\cdot(0,-1,1)$$
$$= (-1,0,1)$$



$$\partial^1(-1,1,2,-1,0) =$$
$$-1\cdot(-1,1,0,0,0)$$
$$+1\cdot(0,-1,1,0,0)$$
$$+2\cdot(0,-1,0,1,0)$$
$$-1\cdot(0,0,0,1,-1)$$
$$+0\cdot(0,1,0,0,-1)$$
$$= (1,-4,1,1,1)$$

This is a <u>linear transformation</u> with a matrix

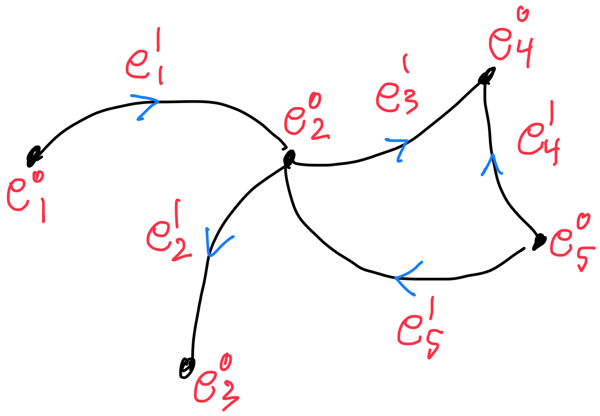
$$D^n = \begin{pmatrix} d^n_{11} & d^n_{12} & \cdots & d^n_{1,c_n} \\ d^n_{21} & \vdots & \ddots & \vdots \\ \vdots & & & \\ d^n_{c_{n-1},1} & \cdots & & d^n_{c_{n-1},c_n} \end{pmatrix} \quad \text{where}$$

d^n_{ij} = i^{th} component of ∂e^n_j
= # of times the i^{th} $(n-1)$-cell appears in the boundary of the j^{th} n-cell, w/ orient.

egs S^1: $\quad D^1 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$

S^2: $\quad D^2 = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \qquad D^1 = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

T^2: $\quad D^2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad D^1 = \begin{pmatrix} 0 & 0 \end{pmatrix}$

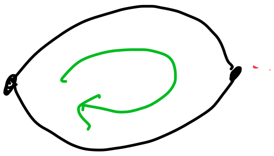


$$D^1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

$$\partial^1(2, 0, \tfrac{3}{2}, -1, \pi) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 3/2 \\ -1 \\ \pi \end{pmatrix} = \begin{pmatrix} -2 \\ \frac{1}{2} + \pi \\ 0 \\ 1/2 \\ 1 - \pi \end{pmatrix}$$

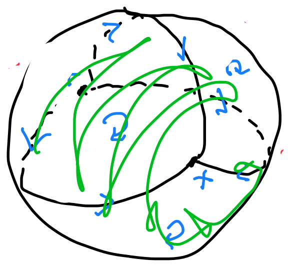
A *cycle* is a chain with empty ($=\vec{0}$) boundary.
These act like the "closed manifolds inside X."

egs



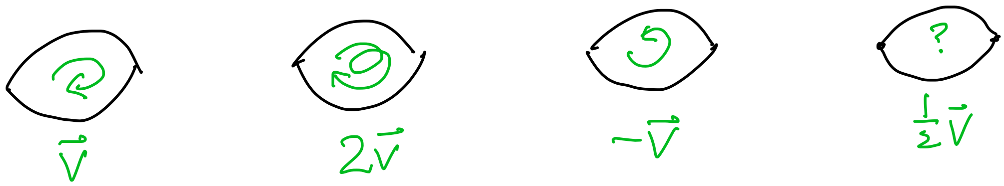
$$\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}\cdot\begin{pmatrix}-1\\ 1\end{pmatrix}=\begin{pmatrix}(-1)(-1)+(-1)\cdot 1\\ 1\cdot(-1)+1\cdot 1\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

cycles



$$\begin{pmatrix}-1 & 0 & 1\\ 1 & -1 & 0\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}=\begin{pmatrix}(-1)+0+1\\ 1+(-1)+0\\ 0+1+(-1)\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

The space of n-cycles is the *null space* of D^n.
Each cycle represents the potential boundary of a hole. But multiples of that cycle "represent the same hole."



...

Z_n = *dimension* of the null space of D^n = nullity (D^n).

egs

(by convention, $Z_0 = C_0$)

S^1: $\quad Z_1 = 1 \quad Z_0 = 2$

S^2: $\quad Z_2 = 1 \quad Z_1 = 2 \quad Z_0 = 2$

T^2: $\quad Z_2 = 1 \quad Z_1 = 2 \quad Z_0 = 1$

The true holes are those that aren't the boundary of some $(n+1)$-chain.

The space of n-cycles that are boundaries is the image of D^{n+1}.

B_{n-1} = dimension of the image of D^n
= $\text{rank}(D^n)$.

egs (by convention, $B_{max} = 0$)

S^1:		$B_1 = 0$	$B_0 = 1$
S^2:	$B_2 = 0$	$B_1 = 2$	$B_0 = 1$
T^2:	$B_2 = 0$	$B_1 = 0$	$B_0 = 0$

Recall: rank + nullity = column dim

$$\boxed{B_{n-1} + Z_n = C_n}$$

<u>Fact</u>: $D^n D^{n+1} = 0$.

"The boundary of a boundary is empty."

<u>Proving</u> this would require a precise definition of CW-complex.

eg. for S^2:

$$D^1 D^2 = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

<u>Corollary</u>: The image of D^{n+1} is contained in the null space of D^n.

$\Longrightarrow \quad B_n \leq Z_n$.

$b_n = n^{th}$ <u>Betti number</u> $= Z_n - B_n \in \mathbb{N}$

$=$ "number of n-dim holes".

$=$ codimension of $\operatorname{im}(D^{n+1})$ in $\operatorname{null}(D^n)$.

egs

S^1: $b_1 = 1 - 0 = 1$

$b_0 = 2 - 1 = 1$

$b_0 = \#$ of connected components

S^1 has a 1-dim hole

S^2: $b_2 = 1 - 0 = 1$

$b_1 = 2 - 2 = 0$

$b_2 = 2 - 1 = 1$

S^2 has a 2-dim hole

T^2: $b_2 = 1 - 0 = 1$

$b_1 = 2 - 0 = 2$

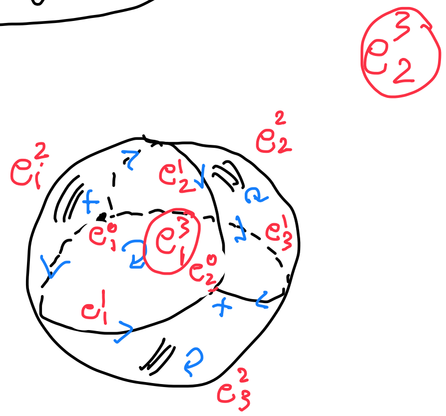
$b_0 = 1 - 0 = 1$

T^2 has a 2-dim hole and <u>two</u> 1-dim holes (meridians)

<u>Fact</u>: The Betti numbers are invariants: they don't depend on the cell structure.

(They are the <u>dimension</u> of the <u>homology groups</u>)

3D eg #1: S^3



$c_0 = 2$

$c_1 = 3$

$c_2 = 3$

$c_3 = 2$

$$D^1 = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \quad D^2 = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \text{ as for } S^2$$

$$D^3 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix}$$

(thumbs point out from e_1^3, in from e_2^3)

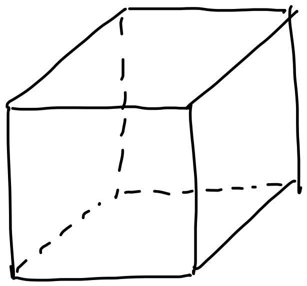
$Z_0 = 2$ $\quad Z_1 = 2$ $\quad Z_2 = 1$ $\quad Z_3 = 1$

$B_0 = 1$ $\quad B_1 = 2$ $\quad B_2 = 1$ $\quad (B_3 = 0)$

$$\boxed{b_0 = 1 \quad b_1 = 0 \quad b_2 = 0 \quad b_3 = 1}$$

"S^3 has one connected component and one 3-dimensional hole."

3D eg #2 : T^3



w/ opposite faces glued.

$c_0 = 1$
$c_1 = 3$
$c_2 = 3$
$c_3 = 1$

$D^1 = (0, 0, 0)$

$D^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$D^3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

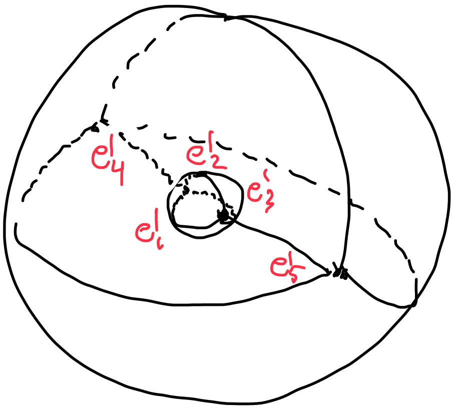
everything appears once + & once − and cancels out.

$Z_0 = 1 \quad Z_1 = 3 \quad Z_2 = 3 \quad Z_3 = 1$

$B_0 = 0 \quad B_1 = 0 \quad B_2 = 0 \quad B_3 = 0$

$\boxed{b_0 = 1 \quad b_1 = 3 \quad b_2 = 3 \quad b_3 = 1.}$

3D eg #3 : $S^2 \times S^1$



$c_0 = 2$

$c_1 = 5$

$c_2 = 6$

$c_3 = 3$

(outer & inner sphere glued.)

$$\begin{array}{c} D^1 = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \\ \qquad\qquad\qquad e_4^1, e_5^1 \text{ loops} \end{array}$$

$$D^2 = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

the two e_1^1's cancel, etc.
we orient things to get 1's.

$$D^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

} these cancel

$Z_0 = 2$ $\quad Z_1 = 4 \overset{2+2}{=}$ $\quad Z_2 = 3 \overset{1+2}{=}$ $\quad Z_3 = 1$

$B_0 = 1$ $\quad B_1 = 3$ $\quad B_2 = 2$ $\quad B_3 = 0$

$\boxed{b_0 = 1 \quad b_1 = 1 \quad b_2 = 1 \quad b_3 = 1}$

Betti #s vs Euler characteristic

Recall $b_n = Z_n - B_n$

$C_n = Z_n + B_{n-1}$. Therefore...

$$\chi = \sum (-1)^n C_n$$
$$= \sum (-1)^n (Z_n + B_{n-1})$$
$$= \sum (-1)^n Z_n + \sum (-1)^n B_{n-1}$$
$$= \sum (-1)^n Z_n - \sum (-1)^n B_n$$
$$= \sum (-1)^n (Z_n - B_n)$$

$$\boxed{\chi = \sum (-1)^n b_n}$$

egs

$$\chi(S^1) = 1 - 1 = 0$$
$$\chi(S^2) = 1 - 0 + 1 = 2$$
$$\chi(T^2) = 1 - 2 + 1 = 0$$
$$\chi(S^3) = 1 - 0 + 0 - 1 = 0$$
$$\chi(T^3) = 1 - 3 + 3 - 1 = 0$$
$$\chi(S^2 \times S^1) = 1 - 1 + 1 - 1 = 0.$$

Invariance

We want to show the Betti numbers are invariant under changing the cell decomposition. Again, being very precise would require defining CW-complexes and other stuff, so I'll make some claims and justify them only with an example.

Let X be a CW-complex as before, with

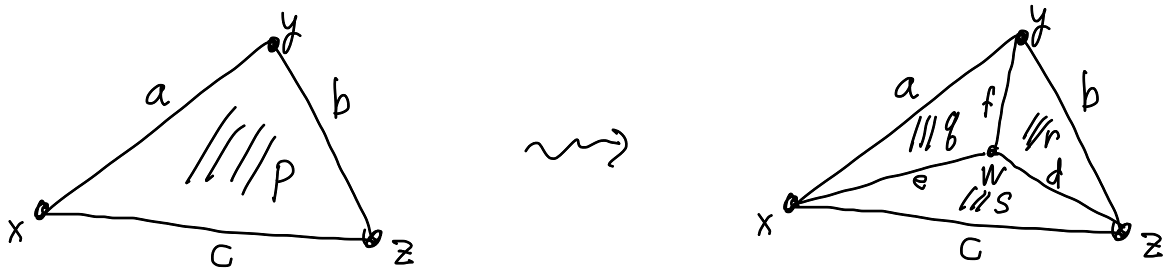
$C_n = \mathbb{R}^{c_n}$ = the vector space of n-chains

$Z_n = \mathrm{Null}(D^n)$ = the subspace of n-cycles

$B_n = \mathrm{Im}(D^{n+1})$ = the subspace of n-boundaries.

Let X' be X with some cells subdivided, and likewise C'_n, Z'_n, B'_n.

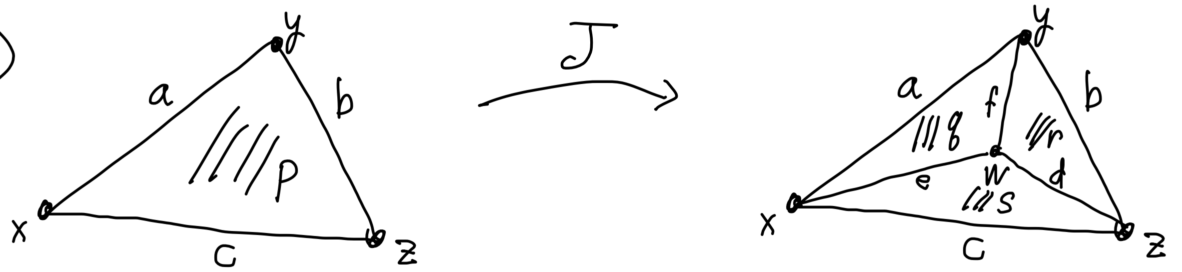
eg



Subdividing a 2-simplex p replaces it by three 2-simplices q, r, s, joined along three new 1-cells d, e, f and a new 0-cell w.

Claim 1: There are linear maps $J^n : C_n \to C'_n$ such that $D^n J^n = J^{n-1} D^n$.

eg



J is the identity on non-subdivided cells

$J(x)=x$, $J(y)=y$, $J(z)=z$, $J(a)=a$, $J(b)=b$, $J(c)=c$

and $J(p) = q+r+s$.

$$DJ(p) = D(q+r+s) = (a+f-e)+(b-f-d)+(e+d-c)$$
$$= a+b-c = JDp.$$

(I haven't drawn the orientation; just trust me. The idea is that d, e, f are in the boundary once w/ each orientation and cancel out.)

We extend J linearly to all chains.

__Claim 2__: There are linear maps $P^n : C'_n \to C_n$ such that $D^n P^n = P^{n-1} D^n$ and $P^n J^n = Id$.

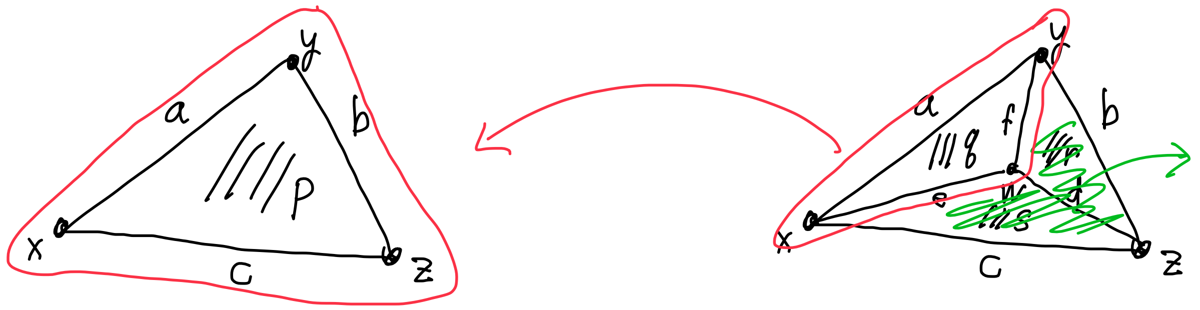
eg Again, P is the identity on non-subdivided cells.

$P(x)=x$, $P(y)=y$, $P(z)=z$, $P(a)=a$, $P(b)=b$, $P(c)=c$

It squashes part of the subdivision, promoting one new cell to be the old one:

$P(w)=z$, $P(d)=0$, $P(e)=c$, $P(f)=b$,

$P(q)=p$, $P(r)=0$, $P(s)=0$.



We again extend P linearly to all chains.

Then $DP(q) = D(p) = a+b-c$

& $PD(q) = P(a+f-e) = P(a)+P(f)-P(e) = a+b-c$

Also $P(J(p)) = P(q+r+s) = p+0+0 = p$.

__Corollary__: $Im(J) \cap B'_n = J(B_n)$.

If Jv is a boundary, then v is a boundary.

Because if $Jv = Du$, then $DPu = PDu = PJv = v$.

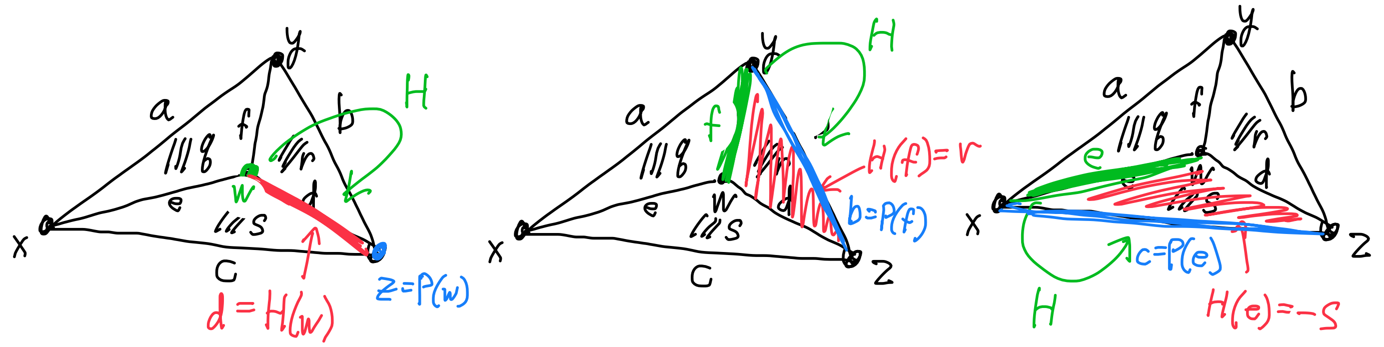
Claim 3: There are linear maps $H^n : C'_n \to C'_{n+1}$

such that $J^n P^n - Id = H^{n-1} D^n + D^{n+1} H^n$.

(called a chain homotopy)

eg $H = 0$ on non-subdivided cells (unlike J & P!)

$H(w) = d$, $H(e) = -s$, $H(f) = r$, $H(d) = 0$



H of a simplex u is a higher-dimensional simplex "connecting" u to $P(u)$.

$(HD+DH)(w) = HD(w) + DH(w) = 0 + D(d) = z - w = JP(w) - w$

$(HD+DH)(f) = H(w-y) + D(r) = (d-0) + (b-f-d) = JP(f) - f$

$(HD+DH)(e) = H(w-x) + D(-s) = (d-0) + (-e-d+c) = JP(e) - e$

Corollary $Z'_n = J(Z_n) + B'_n$.

(= the subspace generated by $J(Z_n)$ and B'_n.)

Because if $v \in Z'_n$, $JP(v) - v = \cancel{HD(v)}^{0} + DH(v)$ (as v is a cycle)

$= DH(v)$

$\Rightarrow v = J(Pv) - D(Hv)$, where $J(Pv) \in J(Z_n)$ and $D(Hv) \in B'_n$.

So we have two subspaces, their intersection, and their sum as subspaces of C'_n:

$$\begin{tikzcd}[column sep=small]
& Z'_n = J(Z_n) + B'_n. & \\
B'_n. \arrow[ur, phantom, "\subseteq" sloped] & & J(Z_n) \arrow[ul, phantom, "\subseteq" sloped] \\
& J(B_n) = B'_n \cap J(Z_n) \arrow[ul, phantom, "\subseteq" sloped] \arrow[ur, phantom, "\subseteq" sloped] &
\end{tikzcd}$$

In this situation we have an "inclusion-exclusion" formula for dimensions:

$$\dim(V+W) = \dim(V) + \dim(W) - \dim(V \cap W)$$

[WARNING: This is not true for >2 subspaces!]

Therefore, we get

$$\dim(Z'_n) = \dim(B'_n) + \dim(J(Z_n)) - \dim(J(B_n))$$

$$\underbrace{\dim(Z'_n) - \dim(B'_n)}_{b'_n} = \underbrace{\dim(J(Z_n)) - \dim(J(B_n))}_{b_n}.$$