Math 361: Real Analysis 2
Assignment # 10

Remember you may use anything we proved in class or previous homework assignments, previous problems in the same homework assignment and even previous parts from the same question even if you did not complete them. The one exception is that if I am asking you to work on part of a proof of a theorem from class you may not use that theorem in your proof (but you may use any part we proved before the part you are proving).

You may also use the major parts of Real I. If you have a question about that please ask.

1. Remember, we defined $\ln(x) = \int_1^x f$. In the last homework you proved that $\ln(x)$ is a strictly increasing, continuous, bijection from $(0, \infty)$ to $\mathbb{R}$. Thus there exists a strictly increasing, continuous, bijection that is the inverse of $\ln(x)$. We will call the function $\exp : \mathbb{R} \to (0, \infty)$.

   Do the following:

   (a) What is $\exp(0)$?

   (b) Show that $\exp \in d((-\infty, \infty))$ and $\exp'(x) = \exp(x)$.

   (c) Explain that $\exp \in c^n((-\infty, \infty))$ for all $n \in \mathbb{N}$, that is show that the function is n times continuously differentiable for all n. A simple explanation suffices, induction seems like overkill.

   (d) Prove for all $a, b \in \mathbb{R}$, $\exp(a + b) = \exp(a)\exp(b)$.

   (e) Show that if $q \in \mathbb{Q}, a \in \mathbb{R}$, $\exp(qa) = \exp(a)^q$.

   (f) Show that if $q \in \mathbb{Q}$, then $\exp(q) = e^q$, where $e = \exp(1)$.

   (g) Show that $2 \leq e$.

2. Let $f, g \in c^2([0, a])$ such that $f(0) = g(0) = f(a) = g(a) = 0$ with g having the property that $g''(x) = -g(x)$. Use integration by parts to show that:

$$\int_0^a fg = -\int_0^a f''g$$

   Notes:

   1. We don't know of any function g that satisfies those conditions. Maybe there isn't one?

3. Remember in a previous homework we defined:

$$I_n = \frac{q^{2n}}{n!} \int_0^a x^n (a - x)^n g(x)\, dx$$

   Where $q > 0$, $a > 0$ and $g \in c([0, a])$ with $g(0) = g(a) = 0$ and $0 < g(x) \leq M$ on $(0, a)$. Let's add to those conditions that $g''(x) = -g(x)$ on $[0, a]$.

   Use the previous problem and results from previous homework to show for $n \geq 2$:

$$I_n = 2q^2(2n - 1)I_{n-1} - q^4 a^2 I_{n-2}$$