Math 361: Real Analysis 2
Assignment # 20

Remember you may use anything we proved in class or previous homework assignments, previous problems in the same homework assignment and even previous parts from the same question even if you did not complete them. The one exception is that if I am asking you to work on part of a proof of a theorem from class you may not use that theorem in your proof (but you may use any part we proved before the part you are proving).

You may also use the major parts of Real I. If you have a question about that please ask.

1. We haven't found things like $\sin(\frac{\pi}{6})$ etc. in our geometry-free definitions of sin and cos. Let's fix that. We have proved lots of things, in particular the formulas: $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$, $\sin(\frac{\pi}{2} - x) = \cos(x)$ might come in handy.
   (a) Show $\cos(2x) = 2\cos^2(x) - 1$ for all x.
   (b) Explain why $\cos(2(\frac{\pi}{3}) + \frac{\pi}{3}) = -1$.
   (c) Expand the above using sum and double angle formulas to get that $\cos(\frac{\pi}{3})$ is a root of polynoimal $4x^3 - 3x + 1 = (x+1)(2x-1)^2$.
   (d) Find $\cos(\frac{\pi}{3})$.
   (e) Find $\sin(\frac{\pi}{3})$.
   (f) Find $\sin(\frac{\pi}{6})$.
   (g) Find $\cos(\frac{\pi}{6})$.

2. (a) Find $\tan(\frac{\pi}{6})$.
   (b) Show:

$$\pi = 2\sqrt{3}\sum_{k=0}^{\infty} \frac{(-1)^k}{3^k(2k+1)}$$

3. In this problem show that for a metric space, the metric definitions of concepts are the same as the topological definition of the concept.
   (a) (Andrej, Lila, Roman) Do this for sequence convergence.
   (b) (Adam, Dylan, Harrison, Lucy) Do this for function continuity.

4. A topological space $(X, \mathcal{T})$ is said to be **Hausdorff space** is for every two different points $a, b \in X$, there exist two disjoint open sets $U, V \in \mathcal{T}$ such that $a \in U$ and $b \in V$.
   (a) Show that in a Hausdorff space that sequence convergence is unique. I.e. show that if $\{a_n\}$ is a sequence in X that converges to L_1 and L_2 then $L_1 = L_2$.
   (b) Show every metric space is a Hausdorff space. Note this show the sequence convergence is unique in metric spaces.