Math 361: Real Analysis 2
Assignment # 21

Remember you may use anything we proved in class or previous homework assignments, previous problems in the same homework assignment and even previous parts from the same question even if you did not complete them. The one exception is that if I am asking you to work on part of a proof of a theorem from class you may not use that theorem in your proof (but you may use any part we proved before the part you are proving).

You may also use the major parts of Real I. If you have a question about that please ask.

1. Let X be a nonempty set and define the discrete metric:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

   Show (X, d) is a metric space.

2. Remember in a topological space (and hence a metric space) a set is closed if and only if its complaint is open. Let (X,d) be a metric space.
   (a) Let $C \subseteq X$. Show that C is closed in (X, d) (i.e. $X \setminus A$ is open) if and only if whenever $\{x_n\} \subseteq C$ and $x_n \to x \in X$, then $x \in C$.
   (b) Show that a closed subset of a complete metric space is complete.
   (c) Show a complete subset of a metric space is closed.

3. Let (X, d) be a metric space. For this problem, if $H \subseteq X$ and $r > 0$ and $a \in H$ then use the notation $b_r^H(a) = \{x \in H : d(a,x) < r\} = b_r^X(a) \cap H$. Let $A \subseteq B \subseteq X$. Also subspaces of $\mathbb{R}$ are assumed to have the usual metric. You may use Heine-Borel on $\mathbb{R}$.
   (a) Show that A is compact when thought of as a subset of B if and only if it is compact when thought of a subset of X.
   (b) Show that A is complete when thought of as a subset of B if and only if it is complete when thought of a subset of X.
   (c) Show $[0, 1)$ is open when thought of as a subset of $[0, 2)$ but not when thought as a subset of $\mathbb{R}$.
   (d) Show $[1, 2)$ is closed when thought of as a subset of $[0, 2)$ but not when thought as a subset of $\mathbb{R}$.
   (e) Is $[1, 2)$ complete?
   (f) Is $[1, 2)$ compact?

4. Suppose that $(X, ||\cdot||)$ is a normed vector space. Show that if $d : X \times X \to \mathbb{R}$ is defined by $d(x,y) = ||x - y||$ then (X, d) is a metric space.