Math 361: Real Analysis 2
Assignment # 22

Remember you may use anything we proved in class or previous homework assignments, previous problems in the same homework assignment and even previous parts from the same question even if you did not complete them. The one exception is that if I am asking you to work on part of a proof of a theorem from class you may not use that theorem in your proof (but you may use any part we proved before the part you are proving).

You may also use the major parts of Real I. If you have a question about that please ask.

1. Let X be as and d_1 and d_2 metrics on X such that there exists $C > 0$ such that for all $x, y \in X$, $d_1(x, y) \leq C d_2(x, y)$.
   (a) Show for all $a \in X, r > 0$, $B^2_{r/C}(a) \subseteq B^1_r(a)$.
   (b) Show if $U \subseteq X$ is open with respect to d_1 then it is open with respect to d_2.

2. Suppose X is a vector space and is a normed vector space with respect to $||\cdot||_a$ and $||\cdot||_b$. We say the two norms are equivalent if there exists $c, C > 0$ such that for all $x \in X$, $c\,||x||_a \leq ||x||_b \leq C\,||x||_a$.
   (a) Show that if $||\cdot||_a$ and $||\cdot||_b$ are equivalent, then the induce the same topological space. I.e. show a set is open in one it is open in the other.
   (b) Show for all $\vec{x} \in \mathbb{R}^n$, $||\vec{x}||_\infty \leq ||\vec{x}||_1 \leq n||\vec{x}||_\infty$.
   (c) Show for all $\vec{x} \in \mathbb{R}^n$, $||\vec{x}||_\infty \leq ||\vec{x}||_2 \leq \sqrt{n}||\vec{x}||_\infty$.
   (d) Show $||\cdot||_1, ||\cdot||_2, ||\cdot||_\infty$ all induce the same topological space on $\mathbb{R}^n$.

3. Let $a \in \mathbb{R}^\infty$ with $\mathbf{a} = \{a_n\}_{n=1}^\infty$
$$a_n = \begin{cases} 5^{-n} & \text{if } n \text{ is odd} \\ 2^{-n} & \text{if } n \text{ is even} \end{cases}$$
   Find:
   1. $||\mathbf{a}||_1$
   2. $||\mathbf{a}||_2$
   3. $||\mathbf{a}||_\infty$

4. Consider $\vec{x} = (1, -2, 0, 1)$ and $\vec{y} = (2, 0, -3, 1)$ to be elements of the inner product space on $\mathbb{R}^4$.
   1. Find $|\langle \vec{x}, \vec{y} \rangle|$.
   2. What upper bound does Cauchy-Schwarz give for this quantity?

5. Consider $f(x) = x$ and $g(x) = \exp(x)$ to be elements of the inner product space $L^2([0, 1])$.
   1. Find $|\langle f, g \rangle|$.
   2. What upper bound does Cauchy-Schwarz give for this quantity?