Math 361: Real Analysis 2
Assignment # 25

Remember you may use anything we proved in class or previous homework assignments, previous problems in the same homework assignment and even previous parts from the same question even if you did not complete them. The one exception is that if I am asking you to work on part of a proof of a theorem from class you may not use that theorem in your proof (but you may use any part we proved before the part you are proving).

You may also use the major parts of Real I. If you have a question about that please ask.

These question are to prepare you for the final.

1. Give an example of each or explain why no such example exists.
    1. A non-continuous, non-monotone function on $[0,1]$ that is in $R[0,1]$.
    2. A function $f \in R[-1,1]$ that has no anti-derivative.
    3. Bounded functions f and g defined on $[0,1]$ with $f \in R[0,1]$ and $g \notin R[0,1]$ such that $f+g \notin R[0,1]$.
    4. A function $f \in R[0,1]$ such that $f(x) \geq 0$ for all $x \in [0,1]$ and $\displaystyle\int_0^1 f = 0$ but $f(c) = 1$ for some $c \in [0,1]$.
    5. A monotone function $f \in R[0,1]$ such that $f(x) \geq 0$ for all $x \in [0,1]$ and $\displaystyle\int_0^1 f = 0$ but $f(c) = 1$ for some $c \in (0,1)$.
    6. A function $f \in c[0,1]$ such that $f(x) \geq 0$ for all $x \in [0,1]$ and $\displaystyle\int_0^1 f = 0$ but $f(c) = 1$ for some $c \in [0,1]$.
    7. a function $f \in R[0,1]$ that is neither monotone nor continuous
    8. a function $f \in R[0,1]$ such that $|f| \notin R[0,1]$
    9. a function $f \notin R[0,1]$ such that $|f| \in R[0,1]$
    10. a sequence of function f_n that converges to a function f pointwise on $[0,1]$ but not uniformly on $[0,1]$
    11. a sequence of function f_n all of which are not continuous, that converges uniformly on $[0,1]$ to a function f that is continuous
    12. a power series with radius of convergence $R = 4$
    13. a power series that converges at $x = 2$ and $x = 4$ but not at $x = 0$
    14. a Hilbert space

2. What is the derivative of $F(x) = \displaystyle\int_x^{x^2} \ln(t)\, dt$ on $[1,e]$?

3. Suppose $f \in R[a,b]$ and $\displaystyle\int_a^b f = -1$. Also suppose there exists $d \in [a,b]$ such that $\int_a^d f = 3$. Show there exists $c \in [d,b]$ such that $\int_a^c f = 0$.

4. Define $f : [0,1] \to \mathbb{R}$ by:

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

Show $f \in R[0,1]$ and compute the value of $\int_0^1 f$.

5. 1. Suppose $\{a_k\}$ and $\{b_k\}$ are sequences such that $|a_k| = |b_k|$ eventually, show that the power series $\sum_{k=0}^{\infty} a_k x^k$ and $\sum_{k=0}^{\infty} b_k x^k$ have the same radius of convergence.
   2. Use the previous part (and series you already know) to show that $\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$ and $\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$ both converge for all x.
   3. Define $\sinh(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$ and $\cosh(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$. Show $f_n(x) = \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!}$ converges uniformly to $\sinh(x)$ on $[-N, N]$ for all $N \in \mathbb{N}$ and $g_n(x) = \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!}$ converges uniformly to $\cosh(x)$ on $[-N, N]$ for all $N \in \mathbb{N}$.
   4. Prove $\frac{d}{dx}\sinh(x) = \cosh(x)$ and $\frac{d}{dx}\cosh(x) = \sinh(x)$.
   5. Let $H(x) = \sinh(x) + \cosh(x)$ show $H'(x) = H(x)$ and $H(0) = 1$. Thus it follows that $H(x) = \exp(x)$.

6. Suppose for all n, f_n is defined on $[0,1]$ by:

$$f_n(x) = \begin{cases} ne^{-nx} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   1. Show $f_n \to f$ pointwise on $[0,1]$ where $f(x) = 0$ for all $x \in [0,1]$.
   2. Compute $\int_0^1 f_n$ for all n.
   3. Does $f_n \to f$ uniformly on $[0,1]$?.

7. Suppose V is a vector space and $||\cdot||$ is a norm on V. Let $a > 0$ and for all $v \in V$ let $||v||_a = a\,||v||$.
   1. Show $||\cdot||_a$ is a norm on V.
   2. Show that if d is the metric derived from $||\cdot||$ then show d_a defined by $d_a(x,y) = a[d(x,y)]$ for all $x, y \in V$ is the metric derived from $||\cdot||_a$.
   3. Show that $\{x_n\} \subseteq V$ and $x \in V$ then $x_n \to x$ using the metric d if and only if $x_n \to x$ using the metric d_a.

8. Find the radius for convergence of $\sum_{n=1}^{\infty} 3^n (x-2)^n$.

9. For all n let $f_n = \sin\left(\frac{x}{n}\right)$ defined on $[0,1]$.
   1. Show $f_n \to 0$ pointwise on $[0,1]$.
   2. Compute f_n'.
   3. Show that $f_n' \to 0$ uniformly on $[0,1]$.
   4. Does $f_n \to 0$ uniformly on $[0,1]$?

5. Find $\int_0^1 f_n$.
6. Use 5 to show that $\left[n - n\cos\left(\frac{1}{n}\right)\right] \to 0$.

10. Consider the metric space $(\mathbb{R}, d)$ where:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}.$$

Does the sequence $\frac{1}{n}$ converge?

11. Let $T : (L_2([1,3]), ||\cdot||_2) \to \mathbb{R}$ defined by $T(f) = 2\int_1^3 f$.

    1. Show $T \in L_2^*([1,3]$.
    2. Find (and prove) $||T||$.

12. Let:

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}.$$

Find:

1. $||A||_{1,1}$
2. $||A||_{2,2}$
3. $||A||_{\infty,\infty}$

13. For each $a \in l_\infty(\mathbb{R})$ define $\mathcal{F}_a : l_1(\mathbb{R}) \to \mathbb{R}$ by $\mathcal{F}_a(b) = \sum_{i=0}^{\infty} a_i b_i$.

    1. Show for each $a \in l_\infty(\mathbb{R})$, $\mathcal{F}_a \in l_1^*(\mathbb{R})$.
    2. Show $F : X \to X^*$ defined by $F(x) = \mathcal{F}_x$ is an isometric embedding of $l_\infty(\mathbb{R})$ into $l_1^*(\mathbb{R})$. (I.e show that it is a norm preserving injection).
    3. Extra credit: Show F is onto so $l_\infty(\mathbb{R}) \cong l_1^*(\mathbb{R})$.

14. Let X be a real inner product space with $< \cdot, \cdot >$. For each $x \in X$ define $\mathcal{F}_x : X \to \mathbb{R}$ by $\mathcal{F}_x(y) =< x, y >$.

    1. Show for each $x \in X$, $\mathcal{F}_x \in X^*$.
    2. Show $F : X \to X^*$ defined by $F(x) = \mathcal{F}_x$ is an isometric embedding of X into X^*. (I.e show that it is a norm preserving injection).
    3. Show that if X is not Hilbert then F is not onto.
    4. Show that if $X = \mathbb{R}^n$ with the usual dot product, then F is onto so $X \cong X^*$.
    5. Extra credit: Show if $X = l_2(\mathbb{R})$ then F is onto so $X \cong X^*$.