Math 361: Real Analysis 2 Assignment # 5

1. In class we defined Thomae's function $h: [0,1] \to \mathbb{R}$ as:

$$h(x) = \begin{cases} 1 & \text{if } x = 0\\ \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ where } \gcd(m, n) = 1\\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

- (a) Show that h is not continuous for all $a \in \mathbb{Q}$.
- (b) Show that h is continuous for all $a \in \mathbb{R} \setminus \mathbb{Q}$.

You may assume:

- 1. Between any two real numbers there exist both a rational and irrational (from Real I).
- 2. For all $\epsilon > 0$, $E_{\epsilon} = \{x \in [0, 1] : h(x) > \epsilon\}$ is finite (from class).
- 2. Suppose that $[c,d] \subseteq [a,b]$. Remember that:

$$\mathbb{1}_{[c,d]}(x) = \begin{cases} 1 & \text{if } x \in [c,d] \\ 0 & \text{if } x \notin [c,d] \end{cases}$$

Show that:

(a)
$$\mathbb{1}_{[c,d]} \in \mathcal{R}[a,b]$$
 and that $\int_{a}^{b} \mathbb{1}_{[c,d]} = d - c.$
(b) $\mathbb{1}_{(c,d]}, \mathbb{1}_{(c,d)}, \mathbb{1}_{[c,d)} \in \mathcal{R}[a,b]$ and that $\int_{a}^{b} \mathbb{1}_{(c,d]} = \int_{a}^{b} \mathbb{1}_{[c,d]} = \int_{a}^{b} \mathbb{1}_{(c,d)} = d - c.$

- 3. Suppose that $f \in c([a, b])$ and further assume $f \in \mathcal{R}[a, b]$ (since we have not yet proved that this is implied but we will). Further assume $f(x) \ge 0$ for all $x \in [a, b]$ and $\int_a^b f = 0$. Show f(x) = 0 for all $x \in [a, b]$
- 4. Find $f \in \mathcal{R}([a,b])$, such that $f(x) \ge 0$ for all $x \in [a,b]$ and $\int_a^b f = 0$, but $f(c) \ne 0$ for some $c \in \mathbb{R}$.