Math 361: Real Analysis 2
Assignment # 6

1. Prove the power rule for derivatives for natural numbers, that is prove $f(x) = x^n$ where $n \in \mathbb{N}$ is differentiable for all $x \in \mathbb{R}$ and $f'(x) = nx^{n-1}$.

   Notes:

   1. For this problem you may use the binomial theorem: for all $a, b \in \mathbb{R}$, $n \in \mathbb{N}$:
   $$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$
   where:
   $$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$
   If you have never proved this before, you should. It is a pretty straightforward induction problem.
   2. Proving the power rule in general (or technically even being able to state it) is still on our to do list for this semester.

2. Let $a \in \mathbb{R}$, $n \in \mathbb{N}$ with $n \geq 2$, $a > 0$ and define $r(x) = (a-x)^n x^n$. Prove:
$$r''(x) = -2n[2n-1](a-x)^{n-1}x^{n-1} + n(n-1)a^2(a-x)^{n-2}x^{n-2}$$
   using the technique of long, painstaking calculation and then massaging it to look like the given form.

   Notes:

   1. I think you all proved the product rule and chain rule for derivatives in real I (but less sure of the power rule hence previous problem) but either way, you may use them.
   2. I recommend you set a time to get together and do it on a giant whiteboard as a group. You can then take pictures of the white board and everyone who worked on it can submit it. No fair submitting pictures if you did not take part in the process. The more people staring at it, the less likely you will make an annoying time-wasting mistake. I don't normally like pictures of a whiteboard for most problems, but long annoying calculations it is okay.
   3. If I am saying it is annoying and the result sure seems extremely uninteresting, why am I having you do it? Well my friends, you are doing the dirty work of digging the foundations of what will by the end of this class be a beautiful piece of architecture.

3. Find $f : [0,1] \to \mathbb{R}$ that is monotone but is discontinuous at infinitely many points.

   Notes:

   1. For this problem, you may use any result that you learned in three semesters of calculus, not just what you have proved in analysis. Because we are going to calculus mode, you don't have to be as precise, just give a good explanation.
   2. We have either proven or are going to prove (depending on when you do this problem) that every function that is either a) continuous at all but finitely many points or b) monotone is Riemann integrable. This example shows that a does not imply b.