Math 361: Real Analysis 2
Assignment # 8

Remember you may use anything we proved in class or the homework. The one exception is that if I am asking you to work on part of a proof of a theorem from class you may not use that theorem in your proof (but you may use any part we proved before the part you are proving). You may also use the major parts of Real I. If you have a question about that please ask.

1. Suppose $f, F : [a, b] \to \mathbb{R}$ with F differentiable and $F'(x) = f(x)$ for all $x \in [a, b]$ and $f \in \mathcal{R}[a, b]$. Suppose $\alpha, \beta \in [a, b]$ show:
$$\int_{\alpha}^{\beta} f = F(\beta) - F(\alpha)$$

2. Suppose $f \in \mathcal{R}[a, b]$ and $m, M \in \mathbb{R}$.
   (a) Suppose $m \leq f(x) \leq M$ for all $x \in [a, b]$ show $m(b-a) \leq \displaystyle\int_a^b f(x) \leq M(b-a)$.
   (b) Suppose $|f(x)| \leq M$ for all $x \in [a, b]$ show $\left|\displaystyle\int_a^b f\right| \leq M(b-a)$
   (c) Suppose $|f(x)| \leq M$ for all $x \in [a, b]$ and $\alpha, \beta \in [a, b]$ show $\left|\displaystyle\int_{\alpha}^{\beta} f\right| \leq M|\beta - \alpha|$

3. Fix $q > 0$, $a > 0$ and $g \in c([0, a])$ with $g(0) = g(a) = 0$ and $0 < g(x) \leq M$ on $(0, a)$. Define:
$$I_n = \frac{q^{2n}}{n!} \int_0^a x^n (a-x)^n g(x)\, dx$$
Show for all $n \geq 0$,
$$0 < I_n \leq \frac{(qa)^{2n}}{n!} aM.$$
Notes:
   1. Remember to explain why that integral even exists.
   2. Remember to show that I_n is strictly positive (hint use a previous homework problem).
   3. Notice that the first part of the integrand is $r_a(x)$ from homework 6. This is the next stage of our super-secret super-cool project. We will see I_n again.

4. For each of the following determine (with proof) if f is Lipschitz continuous on the given domain.
   (a) $f(x) = x^2$ on $[0, 1]$
   (b) $f(x) = x^2$ on $(1, \infty)$
   (c) $f(x) = \sqrt{x}$ on $[1, 2]$
   (d) $f(x) = \sqrt{x}$ on $(1, \infty)$