Math 361: Real Analysis 2
Assignment # 9

Remember you may use anything we proved in class or the homework. The one exception is that if I am asking you to work on part of a proof of a theorem from class you may not use that theorem in your proof (but you may use any part we proved before the part you are proving). You may also use the major parts of Real I. If you have a question about that please ask.

1. Prove the power rule for reciprocal of integers. That is prove if $a > 0$, $n \geq 1$ and $f(x) = x^{1/n}$ $f : [0, \infty) \to [0, \infty)$ is defined by $f(x) = x^{1/n}$ then:

$$f'(x) = \frac{x^{1/n}}{nx} = \frac{1}{n} x^{\frac{1}{n}-1}$$

   Notes:

   1. I assume that you proved for $n \geq 1$, $g : (0, \infty) \to (0, \infty)$ defined by $g(x) = x^n$ is bijective and defined $x^{1/n}$ by $g^{-1}(x) = x^{1/n}$.
   2. $x^{1/n}$ is also known as $\sqrt[n]{x}$.
   3. I put a document about inverse functions on canvas (under files).

2. Let $f(x) = \frac{1}{x}$ on $(0, \infty)$, remember we defined $\ln(x) = \int_1^x f$. In class we showed $\ln(x)$ is differentiable (and hence continuous) on $(0, \infty)$ with its derivative being $\frac{1}{x}$. We also showed that it is strictly increasing and hence injective. Do the following:

   (a) What is $\ln(1)$?

   (b) Prove $\ln(\frac{1}{2}) < 0 < \ln(2)$.

   (c) Prove if $n \in \mathbb{N}$ and $a > 0$ then $\ln(a^n) = n \ln(a)$.

   (d) Show for all $M > 0$ there exists $x > 0$ such that $\ln(x) > M$

   (e) Show for all $m < 0$ there exists $x > 0$ such that $\ln(x) < m$

   (f) Show $\ln : (0, \infty) \to \mathbb{R}$ is bijective.

   (g) Prove that if $a > 0$ then $\ln\left(\frac{1}{a}\right) = -\ln(a)$.

   (h) Prove if $n \in \mathbb{Z}$ and $a > 0$ then $\ln(a^n) = n \ln(a)$.

   (i) Suppose $m \in \mathbb{N}$ and $a > 0$, prove $\ln\left(\sqrt[m]{a}\right) = \frac{\ln(a)}{m}$.

   (j) Let $q \in \mathbb{Q}$ and $a > 0$ show that $\ln(a^q) = q \ln(a)$.

   Notes:

   1. Remember that $a^{\frac{n}{m}} = (a^n)^{\frac{1}{m}} = (a^{\frac{1}{m}})^n$ when $m \in \mathbb{N}$ and $n \in \mathbb{Z}$
   2. I also put up a document about the MVT and its implications on canvas (under files).