Math 14 Survey of Calculus – Exam 2 – Fall 2003

Name: ____

Instructions: **Answer each question completely and show all work.**

1. For each of the following find $f'(x)$. You do not need to simplify your answer.(4 points)

   (a) $f(x) = x^3 + \sqrt{x} + \dfrac{1}{x}$

   $$f'(x) = 3x^2 + \frac{1}{2}x^{-1/2} - \frac{1}{x^2}$$

   (b) $f(x) = (3x+1)\sqrt{x^2-1}$

   $$f'(x) = 3\sqrt{x^2-1} + (3x+1)\frac{1}{2}(x^2-1)^{-1/2}2x = 3\sqrt{x^2-1} + x(3x+1)(x^2-1)^{-1/2}$$

   (c) $f(x) = \dfrac{2x^2}{x^3+1}$

   $$f'(x) = \frac{4x(x^3+1) - 2x^2(3x^2)}{(x^3+1)^2}$$

2. A tire manufacturer has found that the cost of making x tires is given by the function $C(x) = 240 + 64x - .02x^2$ dollars and the revenue received from the sale of x tires is $R(x) = 90x - .03x^2$ dollars.

   (a) What is the marginal profit function

   First notice that the profit function $P(x)$ is given by:

$$P(x) = R(x) - C(x) = 26\,x - 0.01\,x^2 - 240.0$$

   and so the marginal profit function is given by:

$$P'(x) = 26 - 0.02\,x$$

   (b) What is the maximum profit the manufacturer can make selling tires.

   Since $P(x)$ is an upside down parabola it has a unique maximum occurring at its critical point which (solving $P'(x) = 0$) occurs at $x = 1300$. And so the maximum profit is:$16,660.

3. Consider a spherical balloon that is being inflated by helium at the rate of 4 cubic feet per minute. At what rate is the radius increasing when radius is 2 feet. Remember that the volume of a sphere as a function of its radius is given by: $V = \frac{4}{3}\pi r^3$.

We are given that $\frac{dV}{dt} = 4\frac{\text{ft}^3}{\text{min}}$. By differentiating the relationship $V = \frac{4}{3}\pi r^3$ with respect to t we get:

$$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$$

Since at the moment in time that we are concerned with $r = 2\text{ft}$. We get that:

$$4\frac{\text{ft}^3}{\text{min}} = 4\pi(2\text{ft.})^2\frac{dr}{dt}$$

and so:

$$\frac{dr}{dt} = \frac{1}{4\pi}\frac{\text{ft}}{\text{min}}$$

4. Let $f(x) = \dfrac{x^2}{x-1}$

(a) Find the critical points of f.

First we compute:

$$f'(x) = \frac{x\,(x-2)}{(x-1)^2}$$

So $f'(x) = 0$ when $x = 0$ or when $x = 2$, and f' does not exist when $x = 1$. Since the point $x = 1$ is not in the domain of f it is not a critical point, hence the only critical points of f are 0 and 2.

(b) Find the interval(s) where f is increasing and the interval(s) where f is decreasing.

The places where f can change directions is at $x = 0, 1, 2$. Since $f(-1) = \frac{3}{4}$, $f(\frac{1}{2}) = -3$, $f(\frac{3}{2}) = -3$, $f(2) = \frac{3}{4}$, it follows that f is decreasing on $(0,1) \cup (1,2)$ and increasing on $(-\infty, 0) \cup (2, +\infty)$.

(c) Find the relative maximum(s) of f.

From part b we see that a relative maximum occurs at $x = 0$ and so the relative max is $f(0) = 0$

(d) Find the relative minimum(s) of f.

From part b we see that a relative minimum occurs at $x = 2$ and so the relative min is $f(2) = 4$.

5. Let $f(x) = -x^3 + 3x^2 + 24x + 5$.

(a) On what interval(s) is the function concave up?

Since $f''(x) = -6x + 6$ it follows that $f''(x)$ is positive for $x < 1$ and negative for $x > 1$ and so f is concave up on $(-\infty, 1)$

(b) On what interval(s) is the function concave down?

f is concave down on $(1, +\infty)$.

(c) What are the inflection points of f?

The only point of infection is $(1, 31)$.

6. Let $f(x) = x^3 + 3x^2 - 9x + 2$.

(a) Find the maximum value of $f(x)$ on the interval $[-6, 2]$.

First we compute: $f'(x) = 3x^2 + 6x - 9 = 3(x-1)(x+3)$
Thus the critical points of f are $x = 1, x = -3$ and hence the maximum value of the function must occur at any one of the points $-6, -3, 1, 2$. Plugging in these values we get $f(-6) = -52, f(-3) = 29, f(1) = -3, f(2) = 4$ and so the maximum is 29.

(b) Find the maximum value of $f(x)$ on the interval $[-6, 2]$.

From the above the minimum is -52.