Math 14 Survey of Calculus – Exam 3 – Fall 2003

Name: ____

Instructions:**Answer each question completely and show all work.**

1. Solve for x:

   (a) $(3x^4)(-2x^2) = -24$

   $x = \pm\sqrt[3]{2}$

   (b) $\log_x 16 = 2$

   $x = 4$

   (c) $3(e^{4x+1})^2 = 6$

$$\begin{aligned} & 3(e^{4x+1})^2 &= 6 \\ \Rightarrow \quad & (e^{4x+1})^2 &= 2 \\ \Rightarrow \quad & e^{8x+2} &= 2 \\ \Rightarrow \quad & 8x+2 &= \ln(2) \\ \Rightarrow \quad & x &= \tfrac{\ln(2)-2}{8} \end{aligned}$$

   (d) $\log_3(x-6)+\log_3(x-4) = 1$ (Hint: Check that your solutions back in the original equation).

$$\begin{aligned} & \log_3(x-6)+\log_3(x-4) &= 1 \\ \Rightarrow \quad & \log_3((x-6)(x-4)) &= 1 \\ \Rightarrow \quad & (x-6)(x-4) = 3 & \\ \Rightarrow \quad & x^2-10x+24 = 3 & \\ \Rightarrow \quad & x^2-10x+21 = 0 & \\ \Rightarrow \quad & (x-7)(x-3) = 0 & \end{aligned}$$

   This gives the possible solutions $x = 3$ and $x = 7$. However we see that if we plug $x = 3$ back into the original $\log(x-6)$ does not exist. So $x = 7$ is the only solution.

2. For each of the following find $f'(x)$. You do not need to simplify your answer.

(a) $f(x) = e^{3x+2}$

$$f'(x) = 3e^{3x+2}$$

(b) $f(x) = x\ln(x^2) + e^x$

$$f'(x) = x \cdot \frac{2x}{x^2} + \ln(x^2) + e^x = 2 + \ln(x^2) + e^x$$

(c) $f(x) = \dfrac{x^{3/4}e^{(x^2)}}{\sqrt{5x^2+1}}$ (Hint: Let $y = f(x)$ and use logarithmic differentiation)

Let $y = \dfrac{x^{3/4}e^{(x^2)}}{\sqrt{5x^2+1}}$. Then:

$$\ln(y) = \frac{3}{4}\ln(x) + x^2 - \frac{1}{2}\ln(5x^2+1)$$

and so:

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4x} + 2x - \frac{10x}{2(5x^2+1)}.$$

Finally:

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + 2x - \frac{5x}{5x^2+1}\right) = \frac{x^{3/4}e^{(x^2)}}{\sqrt{5x^2+1}}\left(\frac{3}{4x} + 2x - \frac{5x}{5x^2+1}\right).$$

3. Define:

$$g(x) = \frac{\ln x}{x}$$

(a) What is the domain of $g(x)$?

Domain is $(0, +\infty)$ which is the same as the domain of the ln function.

(b) On what intervals is $g(x)$ increasing and on what intervals is $g(x)$ decreasing?

Since:

$$\begin{aligned} g'(x) &= \frac{\frac{1}{x} \cdot x - \ln(x)}{x^2} \\ &= \frac{1 - \ln(x)}{x^2} \end{aligned}$$

and since $x^2 > 0$ for all x, it follows that g is increasing when $1 - \ln(x) > 0$ or when $1 > \ln(x)$ or $x < e$. So the function decreases on $(e, +\infty)$ and increases on $(0, e)$.

(c) Find the absolute maximum and the absolute minimum of $g(x)$ on $[1, 4]$.
We saw from part b that the only critical point for $g(x)$ is at $x = e$. So we need only check $x = 1, e, 4$. So we compute:

$$\begin{aligned} g(1) &= 0 \\ g(e) &= \frac{1}{e} = .3678 \\ g(4) &= \frac{\ln(2)}{2} = .3466 \end{aligned}$$

So the absolute minimum of g is 0 and the absolute maximum is e^{-1}.

4. A town has a population of 10000 people. In 7 years, the population is expected to be 25000 people.

   (a) At this rate what is the expected population in 15 years using the model:

$$Q(t) = Q_0 e^{kt}?$$

   Since $Q(0) = Q_0 = 10000$. So $Q(t) = 10000e^{kt}$. Also $Q(7) = 25000$ so $10000e^{7k} = 25000$ and hence $k = \frac{1}{7}\ln\left(\frac{5}{2}\right)$. Thus $Q(15) \approx 989,900$. So with this model we would expect the population to be around $100,000$ people.

   (b) At this rate what is the expected population in 15 years using the model:

$$Q(t) = \frac{80000}{1 + Be^{-kt}}?$$

   Since $Q_0 = 10000 = \frac{80000}{1+B}$ we know that $B = 7$. Also from $Q(70) = 25000$ we get $k = -\frac{1}{7}\ln\left(\frac{11}{35}\right)$ and thus $Q(15) \approx 50440$. So using this models the expected population is 50440.

5. A radioactive element has a half life of 2100 years. What percentage of the element will be left after 9000 years. (Hint: Radioactive decay follows the model $Q(t) = Q_0 e^{-kt}$.)

Since the half life of the element is 2100 years, this means that $Q(2100) = .5Q_0$, and hence $Q_0 e^{-2100k} = .5Q_0$, and thus $k \approx 0.00033$ plugging this back in we get that $Q(15) = 0.05127Q_0$ and so approximately 5.1% of the element will be left after 9000 years.

6. Suppose that you have a choice of two Investments. Investment A offers a 9.75% return compounded continuously and investment B offers a 10% return compounded semiannually. Which investment has a higher rate of return over a 4 year period?

$$A_{\text{investment A}} = Pe^{.0975 \cdot 4} = 1.47698P$$
$$A_{\text{investment B}} = P(1 + \frac{10}{2})^8 = 1.47745P$$

So investment B is slightly better than investment A.