Math 14 Survey of Calculus – Exam 1 – Fall 2003

Name: ____

Instructions:**Answer each question completely and show all work.**

1. Find the domain of the function $f(x) = \sqrt[4]{x^2 - 4x}$. (4 points)

   Since $f(x)$ exists only when $(x^2 - 4x) \geq 0$, and $(x^2 - 4x) = x(x-4)$ we must have that $x \geq 4$ or $x \leq 0$. This is because when $x \geq 4$ both x and $x - 4$ are both bigger than or equal to 0 and hence the product is bigger than or equal to 0, and when $x \leq 0$ then both x and $x - 4$ are less than or equal to 0 so the product is positive (or 0). Hence:

   $$\text{Domain of } f = (-\infty, 0] \cup [4, +\infty).$$

2. Let $f(x) = 2x^2 + 3$ and $g(x) = 3\sqrt{2x+1}$. For each of the following find rule that defines the function and also give the domain. (5 points each)

   (a) $(f - g)(x)$

   $$\begin{aligned}(f-g)(x) &= f(x) - g(x)\\ &= 2x^2 + 3 - 3\sqrt{2x+1}\end{aligned}$$

   $$\begin{aligned}\text{Domain of } f + g &= \{x : 2x + 1 \geq 0\}\\ &= \{x \geq -\frac{1}{2}\}\\ &= [-\frac{1}{2}, -\infty)\end{aligned}$$

   (b) $(f + g)(x)$

   $$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(3\sqrt{2x+1})\\ &= 2(3\sqrt{2x+1})^2 + 3\\ &= 2(9(2x+1)) + 3\\ &= 36x + 18 + 3\\ &= 36x + 21\end{aligned}$$

   $$\begin{aligned}\text{Domain of } (f \circ g) &= \text{Domain of } g\\ &= [-\frac{1}{2}, -\infty)\end{aligned}$$

3. Let the functions f, g and h be defined by the following graphs:

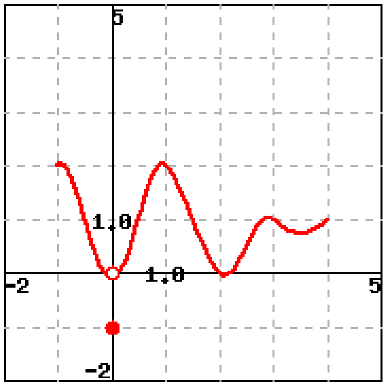

Figure 1: Graph of f

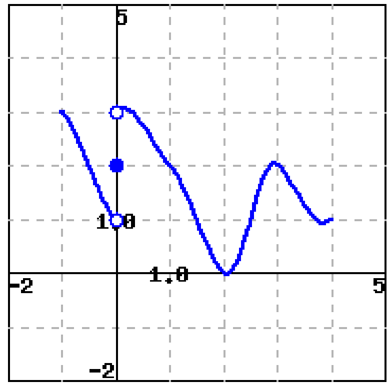

Figure 2: Graph of g

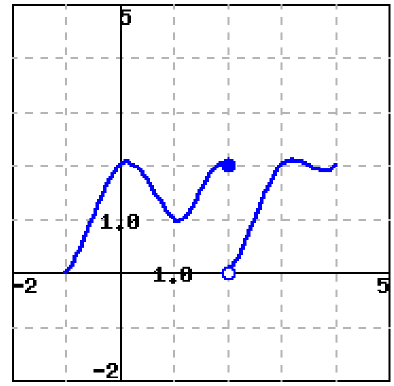

Figure 3: Graph of h

Find the following: (2 points each)

(a) $f(1) = 2$

(b) $(f \circ g)(0) = f(g(0)) = f(2) = 0$

(c) $(f + h)(2) = f(2) + h(2) = 0 + 2 = 2$

(d) $(f \cdot g)(0) = f(0) \cdot g(0) = (-1)(2) = -2$

(e) $\lim_{x \to 2} f(x) = 0$

(f) $\lim_{x \to 0} f(x) = 0$

(g) $\lim_{x \to 0^+} g(x) = 3$

(h) $\lim_{x \to 0^-} g(x) = 1$

(i) $\lim_{x \to 0} g(x) =$DNE

Since the $\lim_{x \to 0^+} g(x) = 1$ and $\lim_{x \to 0^-} g(x) = 3$.

(j) $\lim_{x \to 2} h(x) =$ DNE

Since $\lim_{x \to 2^-} h(x) = 2$ and $\lim_{x \to 2^+} h(x) = 0$

(k) $\lim_{x\to 0}(f+h)(x) = \lim_{x\to 0} f(x) + \lim_{x\to 0} h(x) = 0 + 2 = 2$

(l) $\lim_{x\to 0^+}(g-h)(x) = \lim_{x\to 0^+} g(x) + \lim_{x\to 0^+} h(x) = 3 + 0 = 3$

(m) $\lim_{x\to 0}(f\cdot g)(x) = 0$

Since:

$$\lim_{x\to 0^+}(f\cdot g)(x) = \lim_{x\to 0^+} f(x)\cdot \lim_{x\to 0^+} g(x) = 0\cdot 1 = 0$$

and:

$$\lim_{x\to 0^-}(f\cdot g)(x) = \lim_{x\to 0^-} f(x)\cdot \lim_{x\to 0^-} g(x) = 0\cdot 3 = 0$$

(n) For what values of x on $(-1,4)$ is the function f discontinuous?
The function f is discontinuous at $x=0$ since $0 = \lim_{x\to 0} f(x) \neq f(0) = -1$.

(o) For what values of x on $(-1,4)$ is the function $f\cdot g$ discontinuous?
The function $f\cdot g$ is discontinuous at $x=0$ since $0 = \lim_{x\to 0}(f\cdot g)(x) \neq (f\cdot g)(0) = -2$.

4. Find the following limits, make sure to justify your work using theorems or results from the class: (10 points each)

(a) $\lim_{x\to 2} \dfrac{4x^2-5x-2}{x+1}$
Let:

$$f(x) = \frac{4x^2-5x-2}{x+1}.$$

Since 2 is in the domain of $f(x)$ and f is a rational function it follows that f is continuous at 2.
Hence:

$$\lim_{x\to 2}\frac{4x^2-5x-2}{x+1} = \lim_{x\to 2} f(x) = f(2) = -\frac{4}{3}$$

(b) $\lim_{x\to -1} \dfrac{2x^2-2}{x+1}$

$$\begin{aligned}\lim_{x\to -1}\frac{2x^2-2}{x+1} &= \lim_{x\to -1}\frac{2(x-1)(x+1)}{x+1} \\ &= \lim_{x\to -1} 2(x-1) && \text{since } \tfrac{2(x-1)(x+1)}{x+1} = 2(x-1) \text{ when } x \neq -1 \\ &= 2(-1-1) && \text{since } 2(x-1) \text{ is a polynomial and hence continuous} \\ &= -4\end{aligned}$$

(c) $\lim_{x\to\infty} \dfrac{2x^2-2}{5x^2+1}$

$$\begin{aligned}
\lim_{x\to\infty} \frac{2x^2-2}{5x^2+1} &= \lim_{x\to\infty} \frac{2x^2-2}{5x^2+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{2-\frac{2}{x^2}}{5+\frac{1}{x^2}} \\
&= \frac{\lim_{x\to\infty}(2-\frac{2}{x^2})}{\lim_{x\to\infty}(5+\frac{1}{x^2})} && \text{by limit law} \\
&= \frac{\lim_{x\to\infty} 2 - 2\lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x^2}} && \text{by limit law} \\
&= \tfrac{2-2\cdot 0}{5+0} && \text{since } \lim_{x\to\infty}\frac{1}{x^n} = 0 \text{ if } n>0 \text{ and } \lim_{x\to\infty} c = c \\
&= \tfrac{2}{5}
\end{aligned}$$

5. A function f is defined by

$$f(x) = \begin{cases} 1-x^3+x & \text{if } x<1 \\ x+1 & \text{if } x>1 \\ 12 & \text{if } x=1 \end{cases}$$

Find the following:

(a) $\lim_{x\to 1^-} f(x)$. (3 points)

$$\begin{aligned}
\lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} 1-x^3+x && \text{since } f(x) = 1-x^3+x \text{ when } x<1 \\
&= 1-1^3+1 && \text{since } 1-x^3+x \text{ is a polynomial and hence continuous} \\
&= 1
\end{aligned}$$

(b) $\lim_{x\to 1^+} f(x)$. (3 points)

$$\begin{aligned}
\lim_{x\to 1^+} f(x) &= \lim_{x\to 1^-} x+1 && \text{since } f(x) = x+1 \text{ when } x>1 \\
&= 1+1 && \text{since } x+1 \text{ is a polynomial and hence continuous} \\
&= 2
\end{aligned}$$

(c) $\lim_{x\to 1} f(x)$. (2 points)

Because:

$$1 = \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x) = 2$$

we know that the limit does not exist.

(d) The values of x where $f(x)$ is continuous. (3 points)

From above we know that $\lim_{x\to 1} f(x)$ does not exist it follows that $f(x)$ is not continuous at $x=1$. However the function is continuous everywhere else because both the polynomials $x+1$ and $1-x^3+x$ are continuous.
Hence the function is $f(x)$ is continuous on $(-\infty,1)\cup(1,\infty)$.

6. Let $f(x) = 2x^2 + 1$

(a) Find the slope of the tangent line to the graph of $y = f(x)$ at $x = 1$. (10 points)
We find $f'(x)$ by using the following 4 steps:

$$\begin{aligned}
f(x+h) &= 2(x+h)^2 + 1 = 2(x^2 + 2xh + h^2) + 1 = 2x^2 + 4xh + 2h^2 \\
f(x+h) - f(x) &= 2x^2 + 4xh + 2h^2 - (2x^2 + 1) = 2x^2 + 4xh + 2h^2 - 2x^2 - 1 = 4xh + 2h^2 \\
\frac{f(x+h) - f(x)}{h} &= \frac{4xh + 2h^2}{h} = 4x + 2h \\
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 4x + 2h = 4x
\end{aligned}$$

Thus the slope of the tangent line at $x = 1$ is $f'(2) = 4$

(b) Find the equation of the tangent line to the graph of $y = f(x)$ at $x = 1$. (5 points)
Since the slope of the tangent line is known to be 4 we know the equation for the tangent line is of the form:

$$y = 4x + b$$

and the point (1,f(1)) which is (1,3) is on the line we can solve for b in:

$$3 = 4(1) + b$$

and get that $b = -1$ and so:

$$y = 4x - 1$$

is the equation of the tangent line.