Math 50 Calculus – Exam 2 – Fall 2003

Name: ____

Instructions: **Answer each question completely and show all work.**

1. For each of the following find $f'(x)$. You do not need to simplify your answer.(10 points each)

   (a) $f(x) = x\ln(x) + \sqrt{x} + \dfrac{1}{\sin(x)}$

$$\begin{aligned} f'(x) &= \ln(x) + x \cdot \frac{1}{x} + \frac{1}{2}x^{-1/2} - \frac{1}{(\sin x)^2} \cdot \cos x \\ &= \ln(x) + 1 + \frac{1}{2\sqrt{x}} - \csc x \cot x \end{aligned}$$

   (b) $f(x) = \cos(3^x)$

$$f'(x) = -\sin(3^x) \cdot 3^x \cdot \ln(3)$$

   (c) $f(x) = x^{2\pi e}$

$$f'(x) = (2\pi e)x^{2\pi e - 1}$$

2. Consider a spherical balloon that is being inflated by helium at the rate of 4 cubic feet per minute. (8 points each)

   (a) At what rate is the radius increasing when radius is 2 feet. Remember that the volume of a sphere as a function of its radius is given by: $V = \frac{4}{3}\pi r^3$.

   At every time t we know that:

$$\frac{dV}{dt} = 3\pi r^2 \frac{dr}{dt}.$$

   Thus at the time when the radius is 2 feet we have:

$$4 = 4\pi(2)^2 \frac{dr}{dt}$$

   and so:

$$\frac{dr}{dt} = \frac{1}{4\pi} ft/s$$

   (b) At what rate is the surface area increasing when radius is 2 feet. Remember that the surface area of a sphere as a function of its radius is given by: $SA = 4\pi r^2$.

   At every time t we know that:

$$\frac{dSA}{dt} = 8\pi r \frac{dr}{dt}.$$

   Thus at the time when the radius is 2 feet we have:

$$\begin{aligned} \frac{dSA}{dt} &= 8\pi(2)(\frac{1}{4\pi}) \\ &= 4\pi ft^2/s \end{aligned}$$

3. Suppose a particle travels such that its height at time t is given by: $s(t) = te^t$.(8 points each)

(a) Find the acceleration $a(t)$ as a function of t.

$$v(t) = te^t + e^t = e^t(t+1)$$
$$a(t) = te^t + e^t + e^t = te^t + 2t = e^t(t+2)$$

(b) In what intervals is the particle speeding up and what intervals is the particle slowing down?

$v(t) > 0$ when $t > -1$ and $v(t) < 0$ when $t < -1$. Also $a(t) > 0$ when $t > -2$ and $a(t) < 0$ when $t < -2$. Since the particle is speeding up when $a(t)$ and $v(t)$ have the same sign, the particle is speeding up on $(-\infty, -2) \cup (-1, \infty)$. Since the particle is slowing down when $a(t)$ and $v(t)$ have different signs, the particle is slowing down on $(-2, -1)$.

4. Consider the ellipse defined by the equation: $x^2 + xy + y^2 = 1$.

   (a) Find an expression for $\dfrac{dy}{dx}$ in terms of x and y. (6 points)

   Taking the derivative of both sides with respect to x we get:

$$\begin{array}{rrl} & 2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} & = 0 \\ \Rightarrow & \frac{dy}{dx} & = -\frac{2x+y}{x+2y} \end{array}$$

   (b) Find all points on the ellipse where the tangent line is parallel to the line $y = -x$. (Hint: Two lines are parallel if and only if they have the same slope).(12 points)

   Since we wish to find points where the tangent line has slope equal to -1 (the slope of the line $y = -x$) we set:

$$\begin{array}{rrl} & -1 & = -\frac{2x+y}{x+2y} \\ \Rightarrow & x + 2y & = y + 2x \\ \Rightarrow & y & = x. \end{array}$$

   So we wish to to find points on the ellipse $x^2 + xy + y^2 = 1$ which satisfy $y = x$. So we wish to find solutions to:

$$\begin{array}{rrl} & x^2 + x^2 + x^2 & = 1 \\ \Rightarrow & 3x^2 & = 1 \\ \Rightarrow & x & = \pm\sqrt{\frac{1}{3}} \end{array}$$

   Hence two points on the ellipse have slopes equal to -1:

$$\left(\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right) \text{ and } \left(-\sqrt{\frac{1}{3}}, -\sqrt{\frac{1}{3}}\right)$$

5. The measurement of the radius of a circle is 14 inches, with a possible error of 0.25 inches. Use differentials to approximate the possible error in computing the area of the circle. (10 points)

The area of a circle is given by: $A = \pi r^2$. Using differentials we have:

$$dA = 2\pi r dr.$$

Hence

$$dA = 2\pi(14)(.25).$$

So the approximate error is: $7\pi ft^2$.

6. Prove **from the definition** of the derivative (i.e. do not use any rules of differentiation) that if $f'(4)$ and $g'(4)$ both exists and $s(x) = f(x) + g(x)$ for all x, then:

$$s'(4) = f'(4) + g'(4).$$

(10 points)

By definition of the derivative:

$$\begin{aligned}
s'(4) &= \lim_{h\to 0} \frac{s(4+h) - s(4)}{h} \\
&= \lim_{h\to 0} \frac{f(4+h) + g(4+h) - (f(4) + g(4))}{h} \\
&= \lim_{h\to 0} \frac{f(4+h) + g(4+h) - f(4) - g(4)}{h} \\
&= \lim_{h\to 0} \frac{f(4+h) - f(4) + g(4+h) - g(4)}{h} \\
&= \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} + \frac{g(4+h) - g(4)}{h} \\
&= \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} + \lim_{h\to 0} \frac{g(4+h) - g(4)}{h} \\
&= \lim_{h\to 0} f'(4) + g'(4)
\end{aligned}$$

since $f'(4)$ and $g'(4)$ are known to exist.