Math 50 Calculus – Exam 3 – Fall 2003

Name: ____

Instructions: **Answer each question completely and show all work.**

1. Let $f(x) = 2x^{5/3} - 5x^{4/3}$. Answer each of the following giving exact answers (not decimal approximations). You may use the fact that:

$$f'(x) = \frac{10}{3}x^{1/3}\left(x^{1/3} - 2\right) \quad \text{and} \quad f''(x) = \frac{20(x^{1/3}-1)}{9x^{2/3}}.$$

(a) Find the x and y intercepts of f.

The y-intercept is $f(0) = 0$

$$\begin{aligned} f(x) &= 2x^{5/3} - 5x^{4/3} \\ &= x^{4/3}(2x^{1/3} - 5) \end{aligned}$$

so the x-intercepts are $x = 0$ and $x = \dfrac{125}{8}$.

(b) Find the critical points of f.

Setting $f(x) = 0$ we get critical points of $x = 0$ and $x = 8$.

(c) Find the interval(s) where f is increasing and the interval(s) where f is decreasing.

Since $f' > 0$ when $x < 0$ or when $x > 8$ it follows that f is increasing on $(-\infty, 0) \cup (8, \infty)$ and decreasing on $(0, 8)$.

(d) Find the relative maximum(s) and relative minimum(s)of f.

From the above we see that we have a relative max when $x = 0$ and a relative min when $x = 8$. Plugging these values back into $f(x)$ we get that there is a relative min at $(8, -16)$ and a relative max at $(0, 0)$.

(e) Find the interval(s) where the function concave up and the intervals where it is concave down.

The only places where concavity can change is when f'' is 0 or does not exist. This is true when $x = 0$ or $x = 1$ so by checking sample points we see that f is concave down on $(-\infty, 0) \cup (0, 1)$ and concave up on $(1, +\infty)$.

(f) What are the inflection points of f?

The only infection point is $(1, -3)$.

(g) Use all the information above to sketch the graph of f.

2. Find the following limits. If the limit does not exist, also state whether it is equal to $+\infty$, $-\infty$ or neither. Also in each case give exact answers do not just decimal approximations.

(a) $\lim_{x\to\infty} \frac{\ln(x^2+1)}{x^3+3}$

Since $\lim_{x\to\infty} \ln(x^2+1) = \infty$ and $\lim_{x\to\infty} x^3+3 = \infty$ we can use L'Hospitals Rule to get:

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln(x^2+1)}{x^3+3} &= \lim_{x\to\infty} \frac{\frac{2x}{x^2+1}}{3x^2} \\ &= \lim_{x\to\infty} \frac{2x}{(x^2+1)(3x^2)} \\ &= \lim_{x\to\infty} \frac{2x}{3x^4+3x^2} \\ &= \lim_{x\to\infty} \frac{2}{3x^3+3x} \\ &= \infty\end{aligned}$$

(b) $\lim_{x\to 0} \frac{x^3+2x^2+2x}{\tan^2(x)+3x+1}$

Using continuity we get:

$$\lim_{x\to 0} \frac{x^3+2x^2+2x}{\tan^2(x)+3x+1} = \frac{0}{1} = 0$$

(c) $\lim_{\theta\to\frac{\pi}{2}} \frac{\cos\theta - \theta + \frac{\pi}{2}}{\theta^2 - \frac{\pi^2}{4}}$

Since:

$$\lim_{\theta\to\frac{\pi}{2}} \cos(\theta) - \theta + \frac{\pi}{2} = \lim_{\theta\to\frac{\pi}{2}} \theta^2 - \frac{\pi^2}{4} = 0$$

We can apply L'Hospitals to get:

$$\lim_{\theta\to\frac{\pi}{2}} \frac{\cos\theta - \theta + \frac{\pi}{2}}{\theta^2 - \frac{\pi^2}{4}} = \lim_{\theta\to\frac{\pi}{2}} \frac{-\sin\theta - 1}{2\theta} = \frac{-2}{\pi}$$

(d) $\lim_{x\to\infty}(13x)^{\frac{\ln(9)+1}{\ln(3x)+1}}$

Let $y = \lim_{x\to\infty}(13x)^{\frac{\ln(9)+1}{\ln(3x)+1}}$ so (using L'Hospital's rule):

$$\begin{aligned}
\lim_{x\to\infty}\ln(y) &= \lim_{x\to\infty}\left(\frac{\ln(9)+1}{\ln(3x)+1}\right)\ln(13x)\\
&= (\ln(9)+1)\lim_{x\to\infty}\left(\frac{\ln(13x)}{\ln(3x)+1}\right)\\
&= (\ln(9)+1)\lim_{x\to\infty}\left(\frac{\frac{1}{x}}{\frac{1}{x}}\right)\\
&= (\ln(9)+1)\lim_{x\to\infty}1\\
&= (\ln(9)+1)
\end{aligned}$$

So

$$\begin{aligned}
\lim_{x\to\infty}(13x)^{\frac{\ln(9)+1}{\ln(3x)+1}} &= \lim_{x\to\infty}e^{\ln(y)}\\
&= e^{\lim_{x\to\infty}\ln(y)}\\
&= e^{(\ln(9)+1)}\\
&= 9e
\end{aligned}$$

3. Two stationary patrol cars equipped with radar are 10 miles apart on a highway. As a truck passes the first patrol car, its speed is clocked at 55 miles per hour. Eight minutes later when the truck passes the second patrol car, its speed is clocked at 50 miles per hour. Prove that the truck must have exceeded the speed limit (of 55 miles per hour) at some time during the four minutes. (Hint: Not all the information given will be needed.)

First notice that $8mins$ is $\frac{2}{15}miles$. Denote the time that car passed the first patrol car as $t = 0$ and denote the car's position function by $s(x)$. Then by the mean value theorem (we will assume since we are dealing with a physical process that $s(t)$ is indeed differentiable) there exists a time c in $[0, \frac{2}{15}]$ such that:

$$s'(t) = \frac{s(\frac{2}{15}) - s(0)}{\frac{2}{15} - 0} = \frac{10}{\frac{2}{15}} = 75$$

So at some point the car instantaneous velocity was $75MPH$ and hence the car had to be speeding.

4. The manager of a large apartment complex knows from experience that 100 units will be occupied if the rent is 498 dollars per month. A market survey suggests that, on the average, one additional unit will remain vacant for each 3 dollar increase in rent. Similarly, one additional unit will be occupied for each 3 dollar decrease in rent.

   (a) Find the demand function $p(x)$ where x is the number of units occupied.

   Since $p(100) = 498$ and $p(99) = 501$ and so $p(x) = -3x + 798$.

   (b) What rent should the manager charge to maximize revenue?

   The revenue is given by: $R(x) = xp(x) = -3x^2 + 798x$. Thus the maximum occurs when $R'(x) = 0$ (since the extremes clearly do not maximize the function). Since $R'(x) = -6x + 798$, the maximum occurs at $x = 133$. Hence the manager should charge $p(133) = 399$.

5. Centerville is the headquarters of Greedy Cablevision Inc. The cable company is about to expand service to two nearby towns, Springfield and Shelbyville. There needs to be cable connecting Centerville to both towns. The idea is to save on the cost of cable by arranging the cable in a Y-shaped configuration. Centerville is located at $(9, 0)$ in the xy-plane, Springfield is at $(0, 2)$, and Shelbyville is at $(0, -2)$. The cable runs from Centerville to some point $(x, 0)$ on the x-axis where it splits into two branches going to Springfield and Shelbyville.

   (a) Draw of diagram of the above situation.

   (b) Give the interval that x can lie in.

   x lies in the interval $[0, 9]$.

   (c) Find $L(x)$ the length of the cable for a given choice of x.

   $L(x) = 2\sqrt{x^2 + 4} + 9 - x$.

   (d) Find the location $(x, 0)$ that will minimize the amount of cable between the 3 towns and compute the amount of cable needed. May sure that you check the value of x given is in fact a minimum.

   First we find the critical numbers of $L(x)$. Notice that:

$$L(x) = \frac{2x}{\sqrt{x^2 + 4}} - 1.$$

   Hence the critical numbers for $L(x)$ are when $x = \pm\frac{2}{\sqrt{3}}$. Since $-\frac{2}{\sqrt{3}}$ is not in the domain we need only check $x = 0, 9, -\frac{2}{\sqrt{3}}$. Plugging these values in for x we get:

$$L\left(\frac{2}{\sqrt{3}}\right) = 2\sqrt{3} + 9 = 12.464$$
$$L(9) = 2\sqrt{85} = 18.439$$
$$L(0) = 13$$

   So $x = \frac{2}{\sqrt{3}}$ is the optimal choice.