Math 50 Calculus I– Exam 1 – Fall 2003

Name: ____

Instructions:**Answer each question completely and show all work.**

1. (a) State the precise definition of:

$$\lim_{x\to a} f(x) = L.$$

$\lim_{x\to a} f(x) = L$ if and only if for all $\epsilon > 0$ there exists a $\delta > 0$ such that, if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

(b) Use the above definition to prove that:

$$\lim_{x\to -2}(4 - 2x) = 8.$$

Let $f(x) = -2x + 4$. Fix a number $\epsilon > 0$ and let $\delta = \dfrac{\epsilon}{2}$. Then:

$$\begin{aligned}
&0 < |x - (-2)| < \delta \\
\Rightarrow &|x + 2| < \delta \\
\Rightarrow &|x + 2| < \frac{\epsilon}{2} \\
\Rightarrow &2|x + 2| < \epsilon \\
\Rightarrow &| - 2||x + 2| < \epsilon \\
\Rightarrow &| - 2(x + 2)| < \epsilon \\
\Rightarrow &| - 2x - 4| < \epsilon \\
\Rightarrow &| - 2x + 4 - 8| < \epsilon \\
\Rightarrow &|f(x) - 8| < \epsilon.
\end{aligned}$$

Thus if $0 < |x - (-2)| < \delta$ then $|f(x) - 8| < \epsilon$. Since ϵ is arbitrary, by the definition of the the limit it follows that:

$$\lim_{x\to -2}(4 - 2x) = 8.$$

2. Let the functions f, g and h be defined by the following graphs:

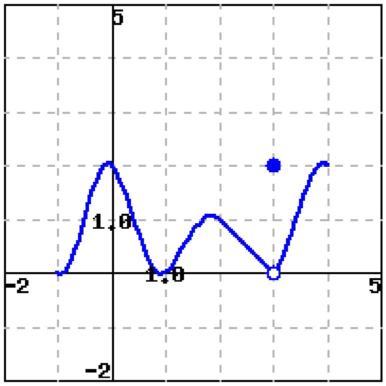

Figure 1: Graph of f

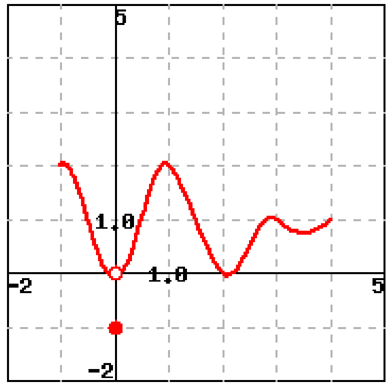

Figure 2: Graph of g

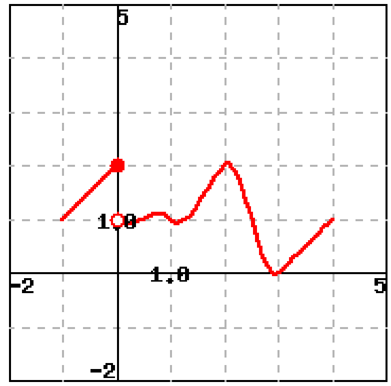

Figure 3: Graph of h

Find the following: (2 points each)

(a) $f(3) = 2$

(b) $(f \circ h)(0) = f(h(0)) = f(2) = 1$

(c) $\lim_{x\to 3} f(x) = 0$

(d) $\lim_{x\to 0^+} h(x) = 1$

(e) $\lim_{x\to 0^-} h(x) = 2$

(f) $\lim_{x\to 0} g(x) =$DNE

Since the
$$\lim_{x\to 0^+} h(x) = 1 \neq \lim_{x\to 0^-} h(x) = 2$$

(g) $\lim_{x\to 3}(h+f)(x) = \lim_{x\to 3} h(x) + \lim_{x\to 3} f(x) = 0 + 0 = 0$

(h) $\lim_{x\to 0^+}(g-h)(x) = \lim_{x\to 0^+} g(x) - \lim_{x\to 0^+} h(x) = 0 - 1 = -1$

(i) $\lim_{x\to 0}(g \cdot h)(x) = 0$

Since:
$$\lim_{x\to 0^+}(g\cdot h)(x) = \lim_{x\to 0^+} g(x) \cdot \lim_{x\to 0^+} h(x) = 0 \cdot 1 = 0$$
and:
$$\lim_{x\to 0^-}(g\cdot h)(x) = \lim_{x\to 0^-} g(x) \cdot \lim_{x\to 0^-} h(x) = 0 \cdot 2 = 0$$

(j) For what values of x on $(-1, 4)$ is the function g discontinuous?
The function g is discontinuous at $x = 0$ since $0 = \lim_{x\to 0} g(x) \neq g(0) = -1$.

3. Find the following limits, make sure to justify your work using theorems or results from the class: (10 points each)

(a) $\displaystyle\lim_{x\to 2} \frac{2x^3 - 10x - 2}{2x+1}$

Let $f(x) = \dfrac{2x^3 - 10x - 2}{2x+1}$ then f is a rational function and hence continuous on its domain. Since $2(2)+1 \neq 0$, 2 is in the domain of f and hence f is continuous at 2 and so:

$$\lim_{x\to 2} \frac{2x^3 - 10x - 2}{2x+1} = f(2) = \frac{2\cdot 8 - 10\cdot 2 - 2}{2\cdot 2 + 1} = -\frac{6}{5}$$

(b) $\displaystyle\lim_{x\to -3} \frac{3x^2 - 12x - 63}{x+3}$

$$\begin{aligned}
\lim_{x\to -3} \frac{3x^2 - 12x - 63}{x+3} &= \lim_{x\to -3} \frac{3(x+3)(x-7)}{x+3} \\
&= \lim_{x\to -3} 3(x-7) && \text{since } \tfrac{3(x+3)(x-7)}{x+3} = 3(x-7) \text{ when } x \neq -3 \\
&= 3(-3-7) && \text{since } 3(x-7) \text{ is a polynomial and hence is continuous at } x=-3 \\
&= -30
\end{aligned}$$

4. Find the following limits. You do not need to justify each step but do show all of your work.

(a) $\displaystyle\lim_{x\to\infty} \frac{4x^3 - 8}{6x^3 + 5x - 1}$

$$\begin{aligned}
\lim_{x\to\infty} \frac{4x^3 - 8}{6x^3 + 5x - 1} &= \lim_{x\to\infty} \frac{4x^3 - 8 \cdot \frac{1}{x^3}}{6x^3 + 5x - 1 \cdot \frac{1}{x^3}} \\
&= \lim_{x\to\infty} \frac{4 - \frac{8}{x^3}}{6 + \frac{5}{x^2} - \frac{1}{x^3}} \\
&= \frac{\lim_{x\to\infty} 4 - \lim_{x\to\infty} \frac{8}{x^3}}{\lim_{x\to\infty} 6 + \lim_{x\to\infty} \frac{5}{x^2} - \lim_{x\to\infty} \frac{1}{x^3}} \\
&= \frac{4-0}{6+0-0} \\
&= \frac{2}{3}
\end{aligned}$$

(b) $\lim_{x\to\infty}\left(\sqrt{4x^2+1}-x\right)$

$$\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{4x^2+1}-x\right) &= \lim_{x\to\infty}\frac{\sqrt{4x^2+1}-x}{1}\\
&= \lim_{x\to\infty}\frac{\sqrt{4x^2+1}-x}{1}\cdot\frac{\sqrt{4x^2+1}+x}{\sqrt{4x^2+1}+x}\\
&= \lim_{x\to\infty}\frac{4x^2+1-x^2}{\sqrt{4x^2+1}+x}\\
&= \lim_{x\to\infty}\frac{3x^2+1}{\sqrt{4x^2+1}+x}\\
&= \lim_{x\to\infty}\frac{3x^2+1}{\sqrt{4x^2+1}+x}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\\
&= \lim_{x\to\infty}\frac{3+\frac{1}{x^2}}{\frac{1}{x^2}\cdot\sqrt{4x^2+1}+\frac{1}{x}}\\
&= \lim_{x\to\infty}\frac{3+\frac{1}{x^2}}{\sqrt{\frac{1}{x^2}\cdot(4x^2+1)}+\frac{1}{x}}\\
&= \lim_{x\to\infty}\frac{3+\frac{1}{x^2}}{\sqrt{4+\frac{1}{x^2}}+\frac{1}{x}}\\
&= \infty
\end{aligned}$$

Since $\lim_{x\to\infty}(3+\frac{1}{x^2})=3$ and $\lim_{x\to\infty}\sqrt{4+\frac{1}{x^2}}+\frac{1}{x}=0+0=0$

(c) $\lim_{\theta\to\frac{\pi}{2}^-}\frac{\ln(\theta)}{\cos(\theta)}$

First notice that both ln and cos are continuous functions and $\frac{\pi}{2}$ is in the domain of both it follows that:

$$\lim_{\theta\to\frac{\pi}{2}^-}\ln(\theta)=\ln(\frac{\pi}{2})>0 \quad \text{since } \frac{\pi}{2}>1$$
$$\lim_{\theta\to\frac{\pi}{2}^-}\cos(\theta)=0$$

Also when values of θ are slighly less then $\frac{\pi}{2}$ then $\cos(\theta)$ is positive (since θ is in the first quadrant). So this give us:

$$\lim_{\theta\to\frac{\pi}{2}^-}\frac{\ln(\theta)}{\cos(\theta)}=+\infty.$$

5. Consider the curve $y = f(t) = t + \frac{1}{t}$ when $t \geq 1$.

(a) Find the slope of the secant line that goes from $t = 2$ to $t = 2.1$. (10 points)

The slope of the secant line from (2,f(2)) to (2.1,f(2.1)) is:

$$\begin{aligned} m &= \frac{f(2.1) - f(2)}{2.1 - 2} \\ &= \frac{2.1 + \frac{1}{2.1} - 2 - \frac{1}{2}}{.1} \\ &= \frac{.1 + \frac{1}{2.1} - \frac{1}{2}}{.1} \\ &= \frac{.32}{.42} \\ &= \frac{32}{42} \\ &= \frac{16}{21} \end{aligned}$$

(b) Find the slope of the tangent line at $t = 2$. (5 points)

The slope of the tangent line at $t = 2$ is equal to the derivative of f at 2. Thus we compute the following:

$$\begin{aligned} f(t+h) &= (t+h) + \frac{1}{t+h} \\ f(t+h) - f(t) &= (t+h) + \frac{1}{t+h} - \left((t) + \frac{1}{t}\right) \\ &= h + \frac{1}{t+h} - \frac{1}{t} \\ &= \frac{ht(t+h) + t - (t+h)}{t(t+h)} \\ &= \frac{ht(t+h) - h}{t(t+h)} \\ &= \frac{h[t(t+h) - 1]}{t(t+h)} \\ \frac{f(t+h) - f(t)}{h} &= \frac{t(t+h) - 1}{t(t+h)} \\ f'(t) &= \lim_{t \to 0} \frac{f(t+h) - f(t)}{h} \\ &= \lim_{t \to 0} \frac{t(t+h) - 1}{t(t+h)} \\ &= \frac{t(t+0) - 1}{t(t+0)} \\ &= \frac{t^2 - 1}{t^2}. \end{aligned}$$

Thus the slope of the tangent line is $f'(2) = \frac{3}{4}$.