Suspension Flows Are Quasigeodesic

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Abstract

A hyperbolic 3-manifold M which fibers over the circle admits a flow called the suspension flow. We show that such a flow can be isotoped to be uniformly quasigeodesic in the hyperbolic metric on M; i.e., the flow lines lifted to hyperbolic space are K-bilipschitz embeddings of $\mathbb{R}$ ($K > 1$ fixed).

Chapter 1

Introduction

A **flow** on a manifold M is a map $\Phi : M \times \mathbb{R} \to M$; if the flow has no stationary points, then the flow lines give an oriented 1-dimensional foliation of M. We write $\Phi_t(x) \equiv \Phi(x,t)$, so that $\Phi_s \circ \Phi_t = \Phi_{s+t}$. For example, a closed 3-manifold has Euler characteristic zero and therefore admits a nowhere zero vector field, which in turn gives a flow given by the integral curves of that vector field. A path $c : \mathbb{R} \to \mathbb{H}^3$ is **quasigeodesic** if there exists $K > 1, L > 0$ such that for all a, b with $d_c(a,b) > L$, we have

$$d_c(a,b) \leq K \cdot d(a,b).$$

Then a flow on a hyperbolic 3-manifold M is **quasigeodesic** if there exist uniform K, L such that every flow line lifts to a is a K, L-quasigeodesic in $\mathbb{H}^3$.

Let M be a hyperbolic 3-manifold which fibers over the circle. Then M can be represented as a product of a hyperbolic surface (the fiber, F) with the unit interval $[0,1]$, with $F \times \{1\}$ glued to $F \times \{0\}$ using a

pseudo-Anosov [Th1] monodromy map Ψ:

$$M = F \times [0,1] \ / \ (x,1) \equiv (\Psi x, 0).$$

Thus M is covered by $F \times \mathbb{R}$. The flow on M obtained by projecting the 1-manifolds $\{x\} \times \mathbb{R}$ from $F \times \mathbb{R}$ to M is called the **suspension flow**, and M is referred to as the **suspension** of F. Our main result is

Main Theorem *The suspension flow on a cusped hyperbolic 3-manifold which fibers over the circle is (can be isotoped to be) uniformly quasi-geodesic.*

Zeghib has shown that a closed hyperbolic 3-manifold does not admit any flow where each flow line is geodesic [Z]; however, there are examples of hyperbolic 3-manifolds which admit quasigeodesic flows. Cannon and Thurston [CT] showed that the suspension flow on any compact hyperbolic 3-manifold fibering over the circle is quasigeodesic. Zeghib [Z] also gave an elementary proof that if M is any compact 3-manifold fibering over the circle, then any flow transverse to the fiber is quasigeodesic. More recently, combined works of Gabai [G], Mosher [Mo1], and Fenley [FM] show that any closed, oriented, hyperbolic three-manifold with nontrivial second homology admits many quasigeodesic flows. In each of the above cases, the manifold under consideration is compact; in this paper, we consider non-compact hyperbolic 3-manifolds which have finitely many cusps.

To prove the main theorem, we make use of the idea of a path "tracking" [F] a geodesic: a path c in $\mathbb{H}^3$ **tracks** a geodesic γ in $\mathbb{H}^3$ if

1. There exists $R > 0$ so that c lies within a neighborhood of radius

R of γ. (c **is close to** γ.)

2. There exists $Q > 0$ such that if the length of a subpath of c with endpoints a and b is at least Q, then the distance between $\pi(a)$ and $\pi(b)$ along γ is at least 1, where π denotes orthogonal projection onto γ (c **makes progress** along γ.)

The following three basic facts are used repeatedly throughout this paper:

Basic Fact 1 *A path in $\mathbb{H}^n$ is quasigeodesic if and only if it tracks a geodesic.*

Basic Fact 2 *Geodesic segments in $\mathbb{H}^n$ with nearby ends are close everywhere.*

Basic Fact 3 *To show that a flow is uniformly quasigeodesic, it suffices to show that all sufficiently long flow lines are quasigeodesic, with a uniform quasigeodesic constant, as we are only interested in the large scale behavior of the flow.*

We will show that the suspension flow on M is uniformly quasigeodesic by showing that there are constants R and Q so that each flow line tracks a geodesic, with these tracking constants.

The manifold M has finitely many cusps. Fix an open neighborhood of each cusp in M; this determines a collection of horoballs in the universal cover $\widetilde{M}$, each a pre-image of a cusp. We may choose these cusp neighborhoods so that each horoball lies a distance at least 1000 from all the other horoballs. Then the **neutered space** N is obtained by removing these open horoballs from $\widetilde{M}$. In this paper, we will sometimes

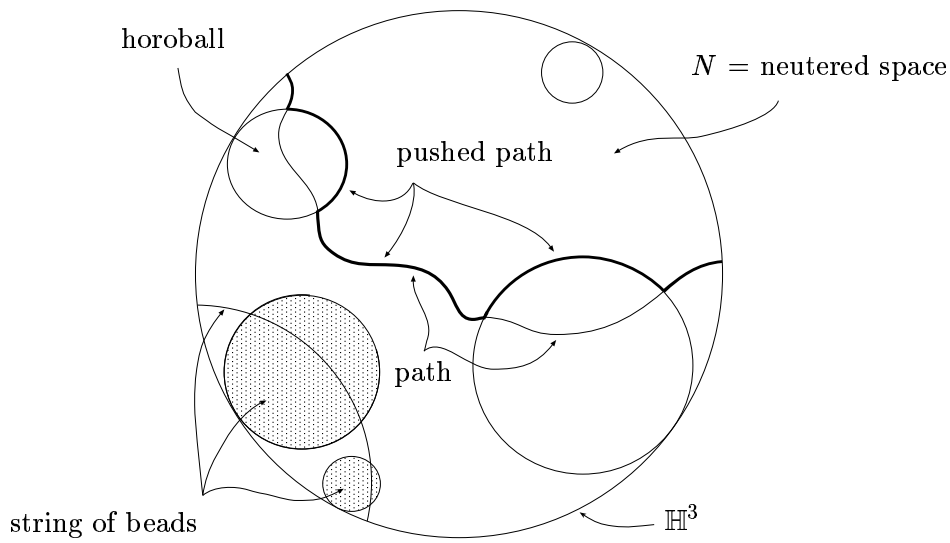

Figure 1.1: A pushed path and a string of beads.

use the term neutered space to also refer to the compact submanifold $M \setminus \{\text{open neighborhood of cusps}\}$. Define $r : \widetilde{M} \to N$ to be the nearest point retraction (in the hyperbolic metric).

In Section 2, the dynamics of the flow which arise from pseudo-Anosov monodromy Ψ are used to prove the following 2 crucial lemmas:

Lemma 1 *Every flow line segment inside a horoball is uniformly quasi-geodesic.*

Lemma 2 *Each flow line enters a given horoball at most once.*

In Section 3, we show that flow lines are close to geodesics (part 1 of the definition of tracking). Given a path c in M, its associated **pushed path** $\overline{c}$ is $r \circ c$ (Figure 1.1). Define the **string of beads** along a geodesic γ in $\mathbb{H}^3$ to be the geodesic γ together with all the horoballs which γ intersects. In section 3.1 we will show that each *pushed* flow

line lies close to the string of beads connecting its endpoints. We prove this by showing

1. Pushed flow lines are uniformly quasigeodesic in the neutered space of M.

2. Any quasigeodesic in the neutered space is close (in the hyperbolic metric) to the string of beads with the same endpoints.

In Section 3.2, we will show that every flow line lies close to a string of beads, as follows: Since a flow line is identical to its corresponding pushed flow line outside the horoballs, the portions of this flow line which lie outside the horoballs are close to the string of beads. The portion of this flow line which lies inside a horoball lies close to the geodesic segment γ' connecting its endpoints by Lemma 1. Since the endpoints of γ' are close to the string of beads, it follows that all of γ' lies close to the string of beads. Thus each flow line is close to a string of beads.

In Section 3.3, we prove that every flow line is close to the *string* in the string of beads. We now know a flow line has segments which lie within a fixed distance, R_{f}, of the string (for which there is nothing to prove) and segments which lie within R_{f} of some bead. The segments which lie within R_{f} of a bead lie completely inside a slightly larger bead (by expanding the bead). Since flow line segments in a horoball are uniformly quasigeodesic, it follows that these flow line segments lie within R_{f} of the string.

Finally, in Section 4 we show that all flow lines make uniform progress along the geodesic with the same endpoints.

Chapter 2

The Dynamics of the Suspension Flow

Cannon and Thurston introduce a useful metric called the singular Solv metric on the suspension X of a closed surface in [CT]. This metric is useful because in the singular Solv metric on the universal cover of X, the lifts of the flow lines of the suspension flow on X are naturally represented as vertical straight lines in $\mathbb{R}^3$. In this section we define a new metric based on the singular Solv metric with this same useful property outside cusp neighborhoods, which will help us understand the dynamics of the flow.

The singular Solv metric is defined as follows. Let $\lambda > 1$ be the stretch factor for the pseudo-Anosov monodromy Ψ, and let $\mathcal{F}^+$ and $\mathcal{F}^-$ be the stable and unstable singular foliations for Ψ. Choose local coordinates (x, y) on F so that the x-axis lies along an unstable leaf and the y-axis lies along a stable leaf. Then the monodromy Ψ expands by the factor λ in the x direction and contracts by the factor $\frac{1}{\lambda}$ in the y

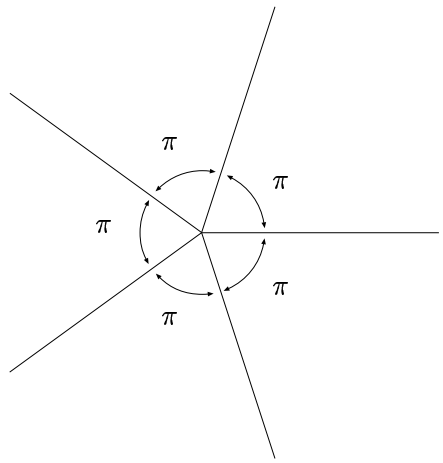

Figure 2.1: A 5-pronged singularity on $\widehat{F}$.

direction.

Away from the singularities, the **singular Solv** metric on $F \times \mathbb{R}$ is just the Solv metric, which is given by the formula

$$ds^2 = \lambda^{2t}dx^2 + \lambda^{-2t}dy^2 + dt^2,$$

where t is the coordinate on $\mathbb{R}$. Note that the singular Solv metric restricted to the xy-plane is the Euclidean metric; this metric is scaled in the x and y directions as the t coordinate changes. In this metric, a rotation by π about the t-axis is an isometry.

If we choose a cylindrical neighborhood of the t-axis in Solv and form the quotient of this neighborhood with rotation by π, we obtain the singular Solv metric with a 1-pronged singularity. Then the singular Solv metric with an n-pronged singularity is obtained by taking the n-fold branched cover of the singular Solv metric with a 1-pronged singularity, branched around the t-axis. Therefore, near an n-pronged singularity, the singular Solv metric is the Euclidean metric with cone angle $n\pi$,

and consists of n copies of a vertical half space in Solv arranged around the t-axis like slices of a cake. We will call each of these half spaces a **section**.

The singular Solv metric on $F \times [0,1]$ is defined so that the map $F \times 1 \to F \times 0$ given by $(w,1) \mapsto (\Psi w, 0)$ is an isometry; thus it induces a singular Solv metric on $M = F \times [0,1] \;/\; (w,1) \equiv (\Psi w, 0)$.

The surface F is a closed surface F^+ minus finitely many points, which gives the manifold M cusp lines; thus our manifold is not compact. However, the suspension M^+ of F^+ is compact. Thinking of M as a subspace of M^+, the restriction of the singular Solv metric puts the cusp lines at a finite distance. Thus the singular Solv metric is not quasi-isometric to the hyperbolic metric on a cusp of M and therefore is not complete. We will modify this metric in neighborhoods of the cusp lines to create a new metric which is quasi-isometric to the hyperbolic metric.

Let $\widehat{F}$ be the cover corresponding to the kernel of the inclusion map $i_* : \pi_1 F \to \pi_1 F^+$. Then $\widehat{F}$ is a punctured plane with a singular Euclidean metric, with cone points where the cone angle is a multiple of π (Figure 2.1). Further, $\widehat{F} \times \mathbb{R}$ is the infinite cyclic cover M_∞ of M.

The manifold M_∞ is a subset of $\widetilde{M^+}$, the universal cover of the closure of M. Let M_∞ inherit the subspace metric of the singular Solv metric on $\widetilde{M^+}$. Define a **cylinder** to be all points (x,y,t) such that the distance from (x,y,t) to the cusp line, in the induced metric coming from the singular Solv metric on the plane $t =$ constant, is at most $\frac{1}{e}$. Then M_∞ is homeomorphic to $\mathbb{R}^3$ with infinitely many cylinders removed. Each of these cylinders covers one of the torus boundary components of N.

When we draw pictures of the Solv metric, we will always draw

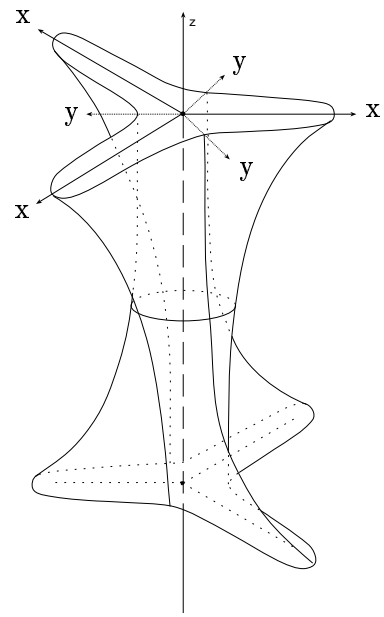

Figure 2.2: A portion of a 3-pronged cylinder of M_∞.

flow lines as vertical straight lines. In this model, a cylinder becomes more eccentrically star shaped, lengthened along the unstable singular foliation and shortened along the stable singular foliation, as t tends to infinity, as in Figure 2.2. The elongating and shortening effect can be more easily seen if we "square off" a 2-pronged cylinder so that it has 4 sides, two of which are parallel to the x-axis and the other two parallel to the y-axis (Figure 2.3).

Choose a cylinder preimage T_∞ in M_∞ of a cusp T with an n-pronged singularity. Choose local cylindrical coordinates (r, θ, t) for T_∞ with $\theta \in [0, n\pi)$, $t \in \mathbb{R}$. Adjust this choice of coordinates if necessary so that $\theta = k\pi$ corresponds to the unstable eigendirections and $\theta = k\pi + \frac{\pi}{2}$ corresponds to the stable eigendirections, with the rest of the values of θ increasing monotonically when traveling counter clockwise from the

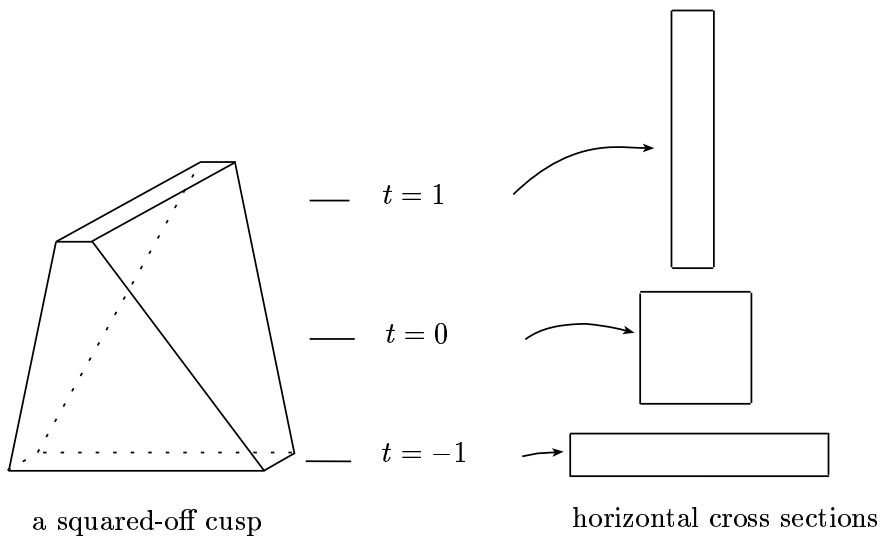

Figure 2.3: A portion of a "squared-off" 2-pronged cusp of M_∞.

$\theta = 0$ eigendirection. Then T_∞ is described by

$$0 < r\sqrt{\lambda^{-2t}\cos^2\theta + \lambda^{2t}\sin^2\theta} \leq \frac{1}{e}.$$

Note that the puncture in each constant t cross section of T_∞ occurs on the t-axis. The universal cover of T_∞ is obtained by unwrapping T_∞ in the θ direction about the t-axis.

Flow lines are vertical in the singular Solv metric on T_∞. Since r, θ, and t are singular Solv coordinates, each flow line which intersects T_∞ is determined by some fixed choice of r and θ, and is parameterized by the coordinate t. If the segment has θ coordinate an integer multiple of $\frac{\pi}{2}$, then it lies in a non-compact leaf of either the stable or unstable 2-dimensional foliation which contains the cusp line. In this case, the flow line segment approaches the cusp line asymptotically in either forwards (i.e. upwards) time (in a stable leaf) or backwards time (in an unstable leaf). We will refer to this type of flow line as a **trapped** flow line, for

one end of the flow line becomes trapped in the cusp forever. If θ is not an integer multiple of $\frac{\pi}{2}$, it is easy to check that this type of flow line will leave the cusp through its boundary in both finite forwards and backwards time. As these flow lines will enter the cusp and then return to the compact part of the manifold, we will call them **returning** flow lines.

2.1 Modifying the Singular Solv Metric

Begin with the singular Solv metric on M. We will re-define this metric in the cusp neighborhoods using the pull-back of the hyperbolic metric under a carefully chosen function f, defined below. We then interpolate between the singular Solv metric on the neutered space of the manifold with the pull-back metric on the cusps along $\mathbb{T}^2 \times I$ product neighborhoods (R-collars). We will call this new metric the **modified metric** on M.

The modification of the metric in T_∞ will be performed by mapping the universal cover $\widetilde{T}$ of T_∞ to the standard horoball $|z| \geq 1$ in the upper half space model of hyperbolic space, then giving $\widetilde{T}$ the pull-back metric. We will define the map f from one *section* of $\widetilde{T}$ to the standard horoball, and then extend this map to the entire horoball by gluing together copies of this section map. Figure 2.4, suggested by Thurston, is a useful picture of the image of one section of the universal cover of T_∞ under this map.

To a first approximation, the map sends (r, θ, t) to rectangular coordinates $(t, \theta, -\ln r)$ in $\mathbb{H}^3$. However, this map must be adjusted so that the boundary of T_∞ is mapped to the plane $z = 1$. Since we have

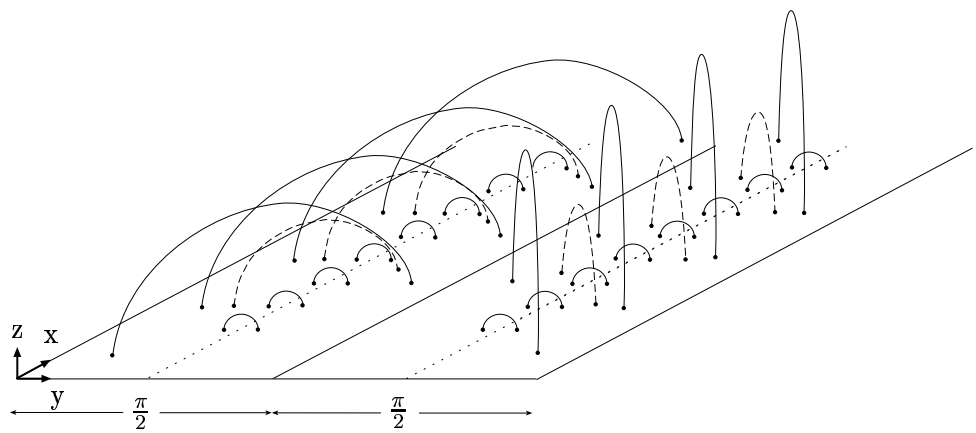

Figure 2.4: One section of the universal cover of a cusp neighborhood.

chosen r so that $r\sqrt{\lambda^{-2t}\cos^2\theta + \lambda^{2t}\sin^2\theta} \leq \frac{1}{e}$, the function f defined by

$$f(r,\theta,t) = (t,\ \tan^{-1}(\lambda^{2t}\tan\theta),\ -\ln(r\sqrt{\lambda^{-2t}\cos^2\theta + \lambda^{2t}\sin^2\theta}))$$

maps the boundary of T_∞ to the plane $z = 1$. Under f, the image of a fundamental domain is $(-\infty,\infty) \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times [1,\infty)$.

Proposition 2.1.1 *The modified metric on M is quasi-isometric to the hyperbolic metric on M.*

Proof: First, any metric on the neutered space N of M is quasi-isometric to the hyperbolic metric on N, by compactness (see Figure 2.5). Furthermore, a cusp under the modified metric is a hyperbolic cusp by the way we construct the modified metric. This cusp will have a different shape than the hyperbolic cusp coming from the actual hyperbolic metric on M. However, we can construct a quasi-isometry between any two hyperbolic cusps as follows. Think of each cusp as the

region of hyperbolic space above the plane $z = 1$. There is a Euclidean linear map A of $\mathbb{R}^3$ which preserves the z coordinate and which takes a fundamental domain for one cusp to a fundamental domain for the other.

Choose two points p and q in the first hyperbolic cusp, and let γ be the geodesic segment connecting them. Then $A(\gamma)$ is a path in the second hyperbolic cusp. The length of this path is an upper bound for the distance between the endpoints $A(p)$ and $A(q)$. The length of any infinitesimal segment of a path is given by its Euclidean length divided by its z-coordinate (which is preserved under A). Further, the scaling in the $x - y$ direction is bounded above by some constant since A is linear. Therefore the distance between $A(p)$ and $A(q)$ in the second cusp is no more than a bounded multiple of the distance between p and q in the first cusp. Similarly, distances in the first cusp are no more than a bounded multiple of distances in the second cusp. This shows that the modified metric on the cusp of M is quasi-isometric to the hyperbolic metric on the cusp. q.e.d.

Therefore the distance between two points measured in the modified metric is within a bounded factor of distance between the same points in the hyperbolic metric, so a quasigeodesic in one metric will necessarily be quasigeodesic (with different quasigeodesic constants) in the other metric. To avoid extra complication we will forget about this quasi-isometry factor, and pretend that distances are the same in two metrics which are quasi-isometric.

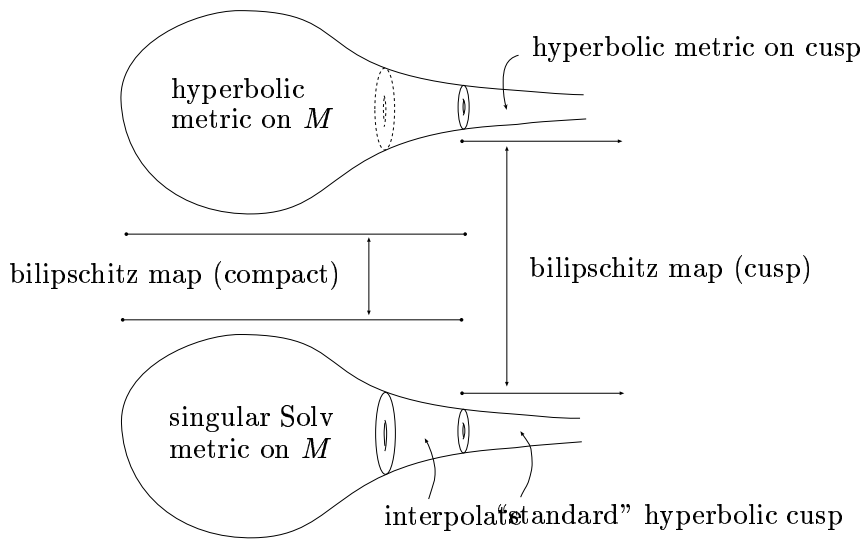

Figure 2.5: Quasi-isometry between modified and hyperbolic metrics.

2.2 The Effect of Pushing a Flow Line

Each flow line travels monotonically in the $\mathbb{R}$ direction in M_∞, typically entering, passing through, and leaving several of the cylinders in M_∞ which cover the boundary of cusps in M. We will push each flow line out of all the cylinders in M_∞ to the boundary of these cylinders (while fixing portions of the flow line which are already outside the cylinders) to produce a pushed flow line in the neutered space. The pushing is realized as follows:

Proposition 2.2.1 *Let ℓ be a flow line, defined by a fixed choice of r and θ, which passes through a cylinder T_∞. Then, in the singular Solv metric, a point (r, θ, t) on ℓ inside T_∞ will be pushed to the point*

$$\left(\frac{1}{e\sqrt{\lambda^{-2t}\cos^2\theta + \lambda^{2t}\sin^2\theta}}, \theta, t \right)$$

on the boundary of T_∞.

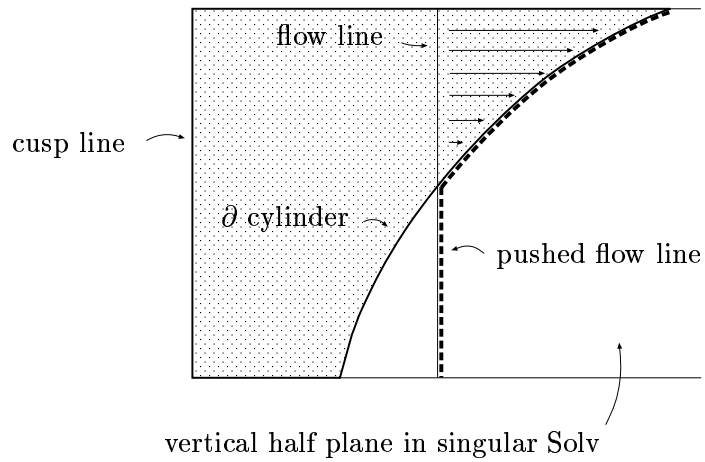

Figure 2.6: Pushing a flow line out of a cusp in the singular Solv metric.

Thus, in the modified singular Solv metric on M_∞, a pushed flow line has constant θ value, the same as that of the unpushed flow line.

Proof: We push a flow line out of the cusp neighborhoods using nearest point retraction in the hyperbolic metric. In the upper half space model, nearest point retraction sends a point (x, y, z) with $z > 1$ to the point $(x, y, 1)$.

The modified singular Solv metric on the cusp neighborhoods is given by the pullback of the map f (page 15). Therefore, the nearest point retraction $(x, y, z) \mapsto (x, y, 1)$ will fix $x = t$ and $y = \tan^{-1}(\lambda^{2t} \tan\theta)$, and will set $z = -\ln(r\sqrt{\lambda^{-2t}\cos^2\theta + \lambda^{2t}\sin^2\theta})$ to 1. Fixing x and y forces t and θ to be fixed, and setting $z = 1$ results in r changing to $1/(e\sqrt{\lambda^{-2t}\cos^2\theta + \lambda^{2t}\sin^2\theta})$. q.e.d.

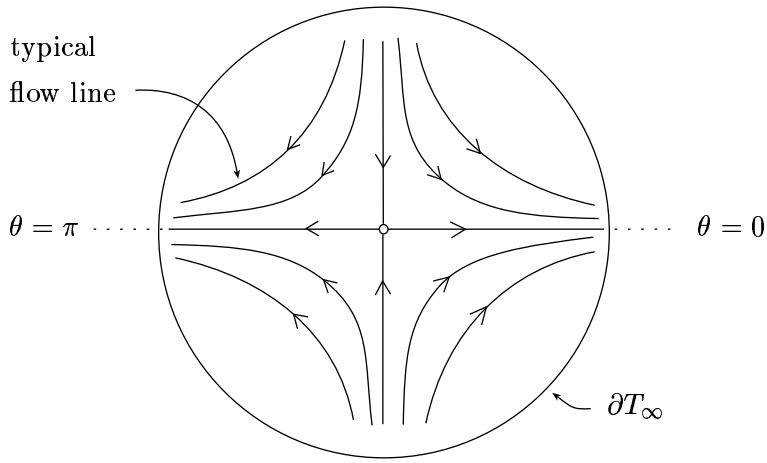

Figure 2.7: Flow lines passing through T_∞, projected onto the plane $t = 0$.

To visualize this pushing, note that a flow line ℓ and a cusp line determine a vertical half plane in M_∞. Then pushing ℓ out of the cylinder containing w is realized by projecting ℓ horizontally across this half plane to the boundary of the cylinder (see Figure 2.6).

Proposition 2.2.2 *Let ℓ be a flow line which passes through a cylinder T_∞. Then, in the singular Solv metric on T_∞, the corresponding pushed flow line rotates around the boundary of T_∞ an angle at most $\frac{\pi}{2}$.*

Proof: Study the dynamics of the flow (Figure 2.7). A flow line moves from the stable direction to the unstable direction, which is an angle $\frac{\pi}{2}$. The corresponding pushed flow line is the original flow line projected to the boundary of this cylinder; thus it rotates around the boundary of T_∞ an angle at most $\frac{\pi}{2}$. q.e.d.

2.3 Proof of Lemma 1

In this section we will prove that every (sufficiently long) segment of flow line inside a horoball is quasigeodesic. Since the modified metric and the hyperbolic metric on M are quasi-isometric, we will show the portions of the flow lines which lie in the horoballs are quasigeodesic in the modified metric by proving that their images are quasigeodesic in $\mathbb{H}^3$ under the map f.

Under this map, the trapped flow lines are Euclidean straight lines going off to infinity in the upper half space model of $\mathbb{H}^3$; these trapped flow lines are shown to be quasigeodesic by a fairly simple calculation. The returning flow line segments are somewhat more complicated to handle: using a direct calculation we will show that if a returning flow segment is sufficiently long, then its length is bounded above by a uniform constant multiple of the hyperbolic distance between its endpoints.

Proposition 2.3.1 *There is a $K_t > 0$ such that all trapped flow line segments are uniformly K_t-quasigeodesic.*

Proof: Suppose ℓ is the image under f of a trapped flow line segment for which $\theta = k\pi$, $k \in \mathbb{Z}$ (the argument in the case $\theta = k\pi + \frac{\pi}{2}$ is similar). Then f simplifies to

$$f(r, \theta, t) = (t, 0, \ln \lambda \cdot t - \ln r),$$

therefore ℓ is a Euclidean straight line segment in the upper half space model of $\mathbb{H}^3$ with slope $\frac{dz}{dx} = \ln \lambda$.

If the endpoints of ℓ are $(x_1, 0, z_1)$ and $(x_2, 0, z_2)$, then an easy cal-

culation shows

$$\begin{aligned}\text{length}(\ell) &= K_{\mathrm{t}} \cdot (\text{length vertical segment between } (x_1, 0, z_1) \text{ and } (x_1, 0, z_2)) \\ &\leq K_{\mathrm{t}} \cdot d_{\mathbb{H}^3}((x_1, 0, z_1), (x_2, 0, z_2)),\end{aligned}$$

where $K_{\mathrm{t}} = \frac{\sqrt{1+(\ln\lambda)^2}}{\ln\lambda}$. Thus such flow line segments are K_{t}-quasigeodesic. q.e.d.

We now consider a segment of a returning flow line inside a horoball. By differentiating the function $f(r, \theta, t)$ in the case where θ is not a multiple of $\frac{\pi}{2}$ one can show that the image of a returning flow line in $\mathbb{H}^3$ travels monotonically in the x and y directions while it is inside the horoball, and increases monotonically in the z direction until it reaches a peak, then decreases monotonically until it exits the horoball again.

Proposition 2.3.2 *There exists a constant K_r depending only on λ such that all (sufficiently long) returning leaf segments are uniformly K_r-quasigeodesic.*

Proof: The segments of flow lines in the cusp have been mapped into hyperbolic space using a carefully defined map f (page 15), which gives us a parametrization of each flow line in the standard hyperbolic cusp. Recall that under the map f, a given flow line is specified by a fixed choice of r and θ, and is parameterized by t. Using f one can calculate that the critical point of each flow line is at $t_{\mathrm{c}} = \frac{1}{2}\log_\lambda \cot\theta$. We define a reparametrization F of f which puts the critical point of each flow line at $t = 0$:

$$F(r, \theta, t) = (t + t_{\mathrm{c}}, \tan^{-1}(\lambda^{2t}), -\ln(r\sqrt{\sin\theta\cos\theta(\lambda^{-2t} + \lambda^{2t})})).$$

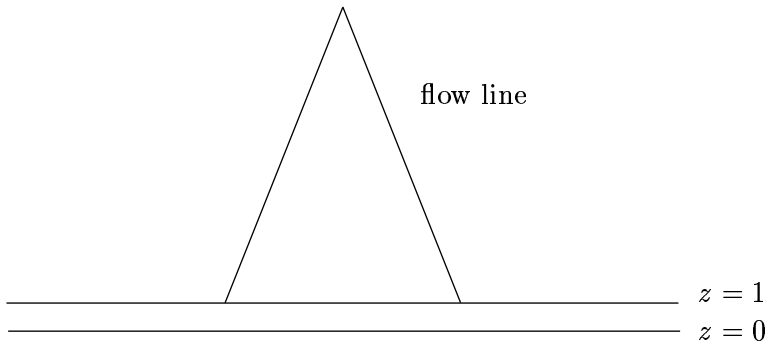

Figure 2.8: An approximation to a long segment of flow line in the cusp.

If $|t|$ is large, then

$$F(r, \theta, t) \approx (t, -\frac{\pi}{2} \text{ or } \frac{\pi}{2}, C_1(\lambda, \theta, r) - C_2(\lambda)|t|),$$

where C_1 and C_2 are constants depending on λ, θ, r and λ respectively. So long segments of flow line in the cusp look like the one in Figure 2.8, which are quasigeodesic.

In the following paragraphs we will use this parameterization F to derive explicit bounds on the behavior of the flow line. These bounds will be put together at the end of the proof to give the result.

Let a and b be the endpoints on the boundary of the horoball of a segment ℓ of returning flow line inside a horoball, and let γ be the geodesic through a and b. Let $|\ |$ denote hyperbolic length, let $|\ |_{\mathrm{E}}$ stand for Euclidean length, and let $z_{\max}$ be the maximum z coordinate of the flow line. Divide the returning flow line segment at half its Euclidean height (at z-value $z_{\mathrm{mid}} = \frac{1}{2}(1 + z_{\max})$) into a top piece, called the **crown**, and two bottom pieces, called **legs** (see Figure 2.9).

The hyperbolic length of the crown is small; using a direct calculation from the map F one can show that in the crown of any flow

line,

$$\begin{aligned}\Delta x &\leq \frac{4+2\ln 2}{\ln\lambda}\cdot z_{\text{mid}}\\ \Delta y &\leq \pi\cdot z_{\text{mid}}\\ \Delta z &\leq 2\cdot z_{\text{mid}}\end{aligned}$$

Therefore, using the taxicab metric, the *hyperbolic* length of the crown of every returning flow line at most $C = \frac{4+2\ln 2}{\ln\lambda} + \pi + 2$ (equation (2.2) at the end of the proof).

Therefore, if a returning flow segment is sufficiently long then the hyperbolic length of the crown is negligible, and so the hyperbolic length of the segment is approximately the sum of the hyperbolic length of its legs. Thus

$$|\ell| \;=\; 2\cdot|\text{leg}| + |\text{crown}| \;\leq\; 2\cdot|\text{leg}| + C \;\approx\; 2\cdot|\text{leg}|,$$

which appears as equations (2.2) and (2.3)) at the end of the proof.

A direct calculation using the explicit parameterization F can be used to show that for sufficiently long returning flow segments (ones with hyperbolic length greater than $\log_\lambda\left(\frac{(\lambda^{-2}+\lambda^2)+\sqrt{(\lambda^{-2}+\lambda^2)^4-16}}{4}\right) - 2\ln\left(\frac{2}{\lambda^{-2}+\lambda^2}\right) + \pi$), the magnitudes of the slopes $\frac{dz}{dx}$ and $\frac{dz}{dy}$ of the legs of the segments are bounded below by uniform constants $B_x = \frac{(\ln\lambda)(\lambda^2-\lambda^{-2})}{\lambda^{-2}+\lambda^2}$ and $B_y = \frac{(\ln\lambda)(\lambda^2-\lambda^{-2})}{2}$. Thus by an argument similar to the one used in the trapped flow line case, the hyperbolic length of one leg is at most a uniform constant multiple K_1 of the length of the vertical segment v from the boundary of the horoball to the top of the leg. The hyperbolic length of v is less than the hyperbolic length of the vertical segment h from the boundary of the horoball to the top of the

flow line segment, because h includes the crown:

$$|\text{leg}| \;\leq\; K_1 \cdot |v| \;<\; K_1 \cdot |h|,$$

as in equations (2.4) & (2.5) at the end of the proof.

The *Euclidean* length of h is equal $z_{\max} - 1$, where $z_{\max}$ is the z coordinate at the top of h. If the returning flow segment is sufficiently long, the Euclidean length of h is approximately $z_{\max}$. A direct calculation using the parameterization F shows that for sufficiently long returning flow segments, $z_{\max}$ is at most a uniform multiple $K_2 = 2 \ln \lambda$ of the Euclidean distance W between a and b. Let D be the vertical segment between the boundary of the horoball and the z-value at the peak of the geodesic segment γ connecting a and b. Then since γ is a semi-circle, W is approximately twice of the Euclidean length of D. Putting this all together, we have

$$|h|_{\mathrm{E}} \;\approx\; z_{\max} \;\leq\; K_2 \cdot W \;\approx\; 2\, K_2 \cdot |D|_{\mathrm{E}},$$

where $|\;|_{\mathrm{E}}$ is Euclidean length. Both h and D are vertical line segments whose base lies on the plane $z = 1$; therefore, taking ln of both sides of the above equation will give the hyperbolic lengths of these two segments:

$$|h| \;\leq\; \ln(2K_2) + |D|,$$

where $|\;|$ denotes hyperbolic length. As K_2 is a constant depending only on λ, for sufficiently long flow lines

$$|h| \;\approx\; |D|,$$

(equation (2.6)). This line segment D is shorter than the geodesic segment connecting a to the top of γ, because D has no horizontal compo-

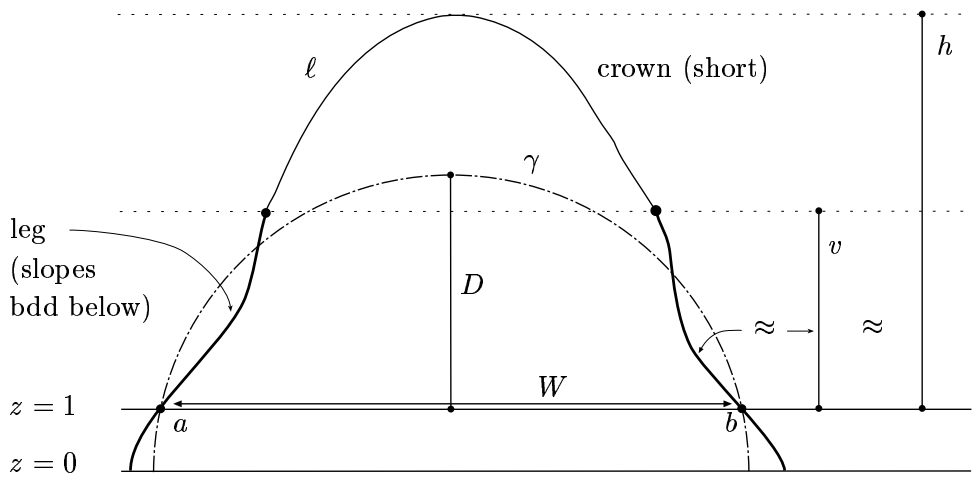

Figure 2.9: A returning flow segment is quasigeodesic.

nent. Thus

$$|D| \leq \frac{1}{2}|\gamma|$$

(equation (2.7)).

Putting all this together, if a returning flow segment is sufficiently long, then

$$\begin{aligned}
|\ell| &= 2 \cdot |\text{leg}| + |\text{crown}| && (2.1)\\
&\leq 2 \cdot |\text{leg}| + C && (2.2)\\
&\approx 2 \cdot |\text{leg}| && (2.3)\\
&\leq 2 \cdot K_1 \cdot |v| && (2.4)\\
&< 2K_1 \cdot |h| && (2.5)\\
&\approx 2K_1 \cdot |D| && (2.6)\\
&< K_1 \cdot |\gamma| && (2.7)\\
&= K_1 \cdot d_{\mathbb{H}^3}(a, b). && (2.8)
\end{aligned}$$

Letting $K_{\mathrm{r}} = K_1$ gives the result. q.e.d.

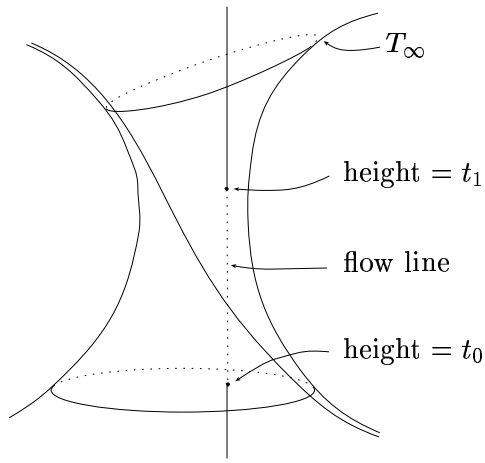

Figure 2.10: A typical flow line intersection with T_∞

Proof: **(Lemma 1)** If $K_c = \max\{K_t, K_r\}$, then every (sufficiently long) segment of flow line inside a horoball is a K_c-quasigeodesic. q.e.d.

2.4 Proof of Lemma 2

In this section we prove Lemma 2, which says that every flow line visits a covering cylinder T_∞ in M_∞ (and hence a horoball in $\widetilde{M}$) at most once.

Proof: [Lemma 2] In singular Solv coordinates, each flow line which intersects T_∞ is determined by some fixed choice of r and θ, and is parameterized by the coordinate t. As flow lines are straight lines in these coordinates, a typical flow line which intersects T_∞ will enter the cylinder at some height t_0 and will remain inside the cylinder until some height $t_1 > t_0$, where it will exit. It is clear from the shape of a cusp

that a flow line may enter any given section of T_∞ at most one time. Every returning flow line remains in the same section of T_∞ as t grows, and every trapped flow line stays on the same line radiating from the origin. Therefore a flow line may visit a covering cylinder T_∞ in M_∞ at most once. Figure 2.10 illustrates this observation in the case that T_∞ 2-pronged cusp. q.e.d.

Chapter 3

Flow Lines Are Near Geodesics

In this section we will find a uniform constant R_Φ such that every flow line segment lies in a neighborhood of radius R_Φ of the geodesic connecting its endpoints. We will establish this result for flow line segments, but we need only pass to longer and longer segments of flow line to have the same result for entire flow lines.

3.1 Pushed Flow Lines Lie Near Strings of Beads

Let $F_N = F \setminus$(open neighborhood of each puncture). The infinite cyclic cover N_∞ of the neutered space N consists of infinitely many copies of $F_N \times [0,1]$ glued together top to bottom using the monodromy map Ψ, so that $N_\infty \simeq F_N \times \mathbb{R}$ (see Figure 3.1). Note $N_\infty \subseteq M_\infty$, where M_∞ is the infinite cyclic cover of M.

3.1.1 Pushed Flow Segments are Quasigeodesic in N

Proposition 3.1.1 *There exists constants $K_p, L_p > 0$ such that every pushed flow line segment of length at least L_p is a K_p-quasigeodesic in the inherited metric on N_∞, and hence a K_p-quasigeodesic in N.*

Proof: We want to compare the distance along a pushed flow line between two points with the geodesic distance between the two points. To do this, we show an even stronger result: that the maximum length of a pushed flow line between successive lifts of F is no more than a bounded multiple of the minimum (neutered space) distance between the lifts.

Let π_2 be the map $\pi_2 : N_\infty \simeq F_N \times \mathbb{R} \to \mathbb{R}$ given by projection onto the second coordinate. Then π_2 restricted to any pushed flow line is monotone, so once any pushed flow line passes through one copy of the fiber $F_N \times \{t\}$ for $t \in \mathbb{R}$, it never returns. Thus, the length of any given pushed flow line segment is no more than the maximum length of a pushed flow line segment between $F_N \times \{0\}$ and $F_N \times \{1\}$ multiplied by the number n of copies of $F_N \times [0,1]$ the pushed flow line intersects.

Each flow line travels distance 1 (in the singular Solv metric) between $F \times \{0\}$ and $F \times \{1\}$. Since we have chosen our horoballs to have a distance at least 1000 from each other, a flow line will encounter at most one covering cylinder T_∞ in $F \times [0,1]$.

Consider the *Euclidean* metric on the intersection $T_{[0,1]}$ of such a cylinder T_∞ with $F \times [0,1]$. A pushed flow line travels distance 1 in the I direction and rotates an angle at most $\pi/2$ around the boundary of the cylinder by Proposition 2.2.2; in other words, at most half way around

the cylinder (in the case of a 1-pronged singularity). Since a Euclidean cylinder has circumference $\frac{2\pi}{e}$, the Euclidean length of each pushed flow line segment in $T_{[0,1]}$ is bounded above by the constant $E = 1 + \frac{\pi}{e}$. The singular Solv metric on $T_{[0,1]}$ is K-quasi-isometric (for some $K > 0$) to the Euclidean metric since $T_{[0,1]}$ is compact; therefore, the singular Solv length of each pushed flow line segment in $T_{[0,1]}$, and therefore in $F_N \times [0,1]$, is bounded above by the constant $A = KE$.

Let ϵ be the minimum distance the compact surfaces $F_N \times \{0\}$ and $F_N \times \{1\}$ in N_∞. Choose $K_{\mathrm{p}} > \frac{2A}{\epsilon}$ and $L_p = A$. Let $\overline{\ell}$ be a finite segment of pushed flow line with endpoints a and b and with length at least L_p. Then an easy calculation shows that

$$d_{\overline{\ell}}(a,b) \;\leq\; K_{\mathrm{p}} \cdot \epsilon \;\leq\; K_{\mathrm{p}} \cdot d_{N_\infty}(a,b).$$

Therefore every pushed flow line of length at least L_{p} is a K_{p}-quasigeodesic. q.e.d.

3.1.2 Quasigeodesic Segments Lie Near Strings of Beads

Let $CH(\gamma_+)$ denote the convex hull of the string of beads γ_+, given by the union of all the tetrahedra with all four vertices in γ_+, and let $\pi_{\mathrm{CH}} : \widetilde{N} \to CH(\gamma_+)$ be nearest point retraction onto this convex hull. The following 3 lemmas will be used to show that any quasigeodesic segment must lie close to the string of beads connecting its endpoints. Lemma 3 is a standard fact from hyperbolic geometry, while Lemmas 4 and 5 are straightforward applications of the thin triangles property of hyperbolic space (Lemma 5 follows from Lemma 4).

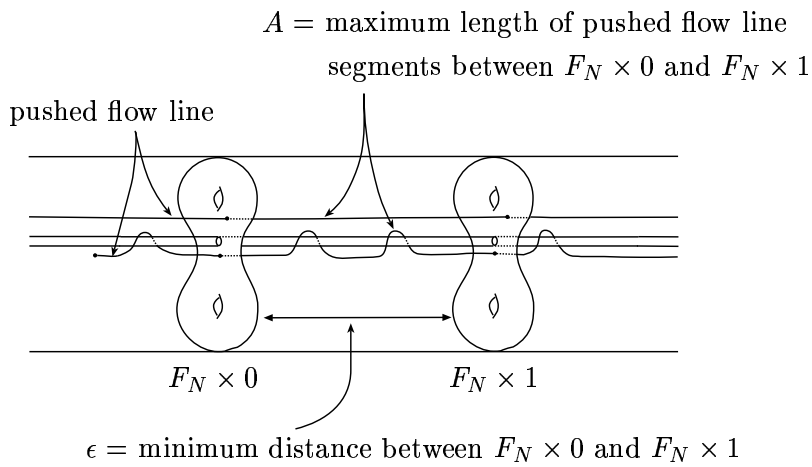

Figure 3.1: Distance comparison in the infinite cyclic cover of N.

Lemma 3 *If c is a path segment which is a hyperbolic distance at least r from $CH(\gamma_+)$, then $|c| \geq \sinh r \cdot |\pi_{CH}(c)|$.*

Lemma 4 *Every point in a geodesic tetrahedra in hyperbolic space lies with distance 2δ of an edge, where δ is the thin triangles constant* $\ln(1+\sqrt{2})$.

Lemma 5 *Every point in $CH(\gamma_+)$ lies within distance 4δ of γ_+.*

We will use the Lemma 3 to show that a pushed flow line segment cannot stray very far away from the string of beads connecting its endpoints, for if it does, there exists a path in N with the same endpoints which is much shorter than the pushed flow line segment, contradicting that the pushed flow line is quasigeodesic in N.

Proposition 3.1.2 *Given $K > 1$, there exists $R > 0$ such that if q is any K-quasigeodesic segment in N and γ is the hyperbolic geodesic seg-*

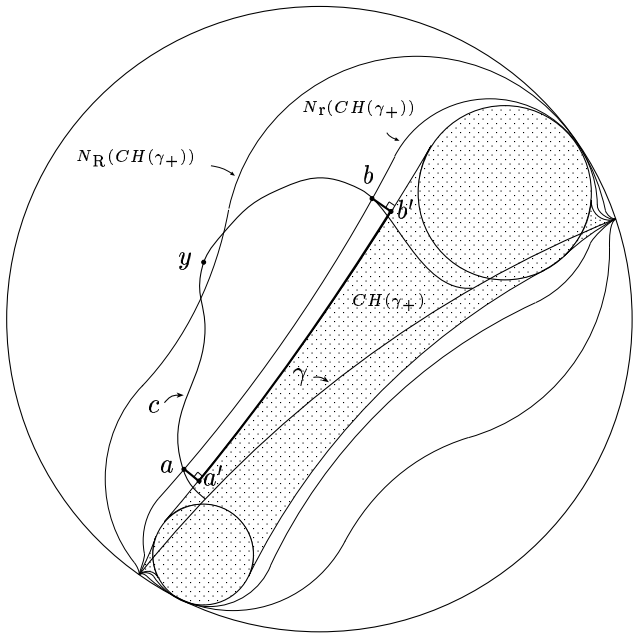

Figure 3.2: Quasigeodesic segments lie near convex hulls of strings of beads

ment connecting the endpoints of q, then q lies within hyperbolic distance R of the convex hull of the string of beads along γ.

Proof: We will show that if a neutered space quasigeodesic gets too far from the convex hull of the string of beads connecting its endpoints, then projecting the path onto the convex hull (followed by pushing the projected path out of the horoballs) produces a much shorter path in the neutered space, contradicting that the original path was quasigeodesic.

Choose $r > \sinh^{-1}(2Ke^{4\delta})$ and let $R = 2Kre^{r+4\delta} + r$, where δ is the thin triangles constant. Then q lies completely inside the hyperbolic R-neighborhood of the convex hull of the string of beads along γ, for

suppose a point y in q falls outside this neighborhood. Then there is a subsegment c of q (containing y), which is at least hyperbolic distance r from the convex hull of the string of beads, of hyperbolic length at least

$$|c| \quad \geq \quad 2(R-r) = 4Kre^{r+4\delta} \tag{3.1}$$

(see Figure 3.2). Let a and b be the endpoints of this subsegment c, and let s be the path $A \cup \pi_{\text{CH}}(c) \cup B$, where A and B are paths of hyperbolic length r connecting a to $a' = \pi_{\text{CH}}(a)$ and b to $b' = \pi_{\text{CH}}(b)$, respectively. Paths A, B, and $\pi_{\text{CH}}(c)$ need not be in $\widetilde{N}$, but may pass through many horoballs not on the string of beads. Let $\overline{A}$, $\overline{B}$, and $\overline{\pi_{\text{CH}}(c)}$ be the results of pushing A, B, and $\pi_{\text{CH}}(c)$ out of the horoballs into $\widetilde{N}$.

The path $\pi_{\text{CH}}(c)$ lies along the boundary of the convex hull of the string of beads γ_+. Let H be a horoball that $\pi_{\text{CH}}(c)$ enters. Note that H cannot be a bead along γ_+ since H contains $\pi_{\text{CH}}(c)$. By Lemma 5, every point in $H \cap CH(\gamma_+)$ lies within hyperbolic distance 4δ of γ_+. But by our initial choice of horoballs, H lies a distance at least 1000 from all the other horoballs. Thus $H \cap CH(\gamma_+)$ lies with hyperbolic distance 4δ of γ. So no point of $CH(\gamma_+)$ inside H can be further than 4δ from the boundary of H. Therefore, pushing $\pi_{\text{CH}}(c)$ from the interior of H onto the boundary of H does not increase its length by more than $e^{4\delta}$:

$$|\overline{\pi_{\text{CH}}(c)}| \quad \leq \quad e^{4\delta} \cdot |\pi_{\text{CH}}(c)|. \tag{3.2}$$

A similar argument shows that each of $\overline{A}$ and $\overline{B}$ will be no longer than $e^{r+4\delta} \cdot r$. Therefore

$$
\begin{aligned}
K \cdot |\overline{s}| &= K \cdot |\overline{A} \cup \overline{\pi_{\mathrm{CH}}(c)} \cup \overline{B}| \\
&\leq K \cdot (e^{r+4\delta} \cdot r + e^{4\delta} \cdot |\pi_{\mathrm{CH}}(c)| + e^{r+4\delta} \cdot r) && \text{by Equation 3.2} \\
&= 2Kre^{r+4\delta} + Ke^{4\delta} \cdot |\pi_{\mathrm{CH}}(c)| \\
&\leq \tfrac{|c|}{2} + Ke^{4\delta} \cdot |\pi_{\mathrm{CH}}(c)| && \text{by Equation 3.1} \\
&< \tfrac{|c|}{2} + \tfrac{Ke^{4\delta}}{\sinh r}|c|) && \text{by Lemma 3} \\
&= \tfrac{|c|}{2} + \tfrac{Ke^{4\delta}}{2Ke^{4\delta}}|c| \\
&\leq \tfrac{|c|}{2} + \tfrac{|c|}{2} \\
&= |c|,
\end{aligned}
$$

Thus $\overline{s}$ has length less than $\frac{1}{K}$ times the length of c, contradicting that $c \subseteq q$ is a K-quasigeodesic. q.e.d.

Proposition 3.1.3 *There exists $R_s > 0$ such that every pushed flow line segment lies within R_s of the string of beads connecting its endpoints.*

Proof: Since pushed flow lines are K_{p}-quasigeodesics, by the previous proposition there exists a constant $R_{\mathrm{p}} > 0$ so that every pushed flow line segment lies within a neighborhood of radius R of the *convex hull* of the string of beads connecting its endpoints. Thus, in light of Lemma 5, every pushed flow line segment lies within $R_{\mathrm{s}} = Rp + 4\delta$ of the string of beads connecting its endpoints. q.e.d.

3.2 Flow Segments Lie Near Strings of Beads

We now know that a pushed flow line segment lies in a neighborhood of the string of beads along the geodesic connecting its endpoints. To prove that flow line segments lie near the string of beads connecting their endpoints, we only need to prove that the segments of flow line which lie inside a horoball lie in a uniform neighborhood of the string of beads. This does not follow automatically from what we have already done, for given a string of beads, most horoballs are not beads on this string. A segment of flow line which lies inside one of the beads trivially lies inside a neighborhood of the string of beads, but the non-bead horoballs need special consideration.

Proposition 3.2.1 *Every flow line segment lies inside a neighborhood of radius R_f of the string of beads along the geodesic connecting its endpoints.*

Proof: Let ℓ be a flow line segment. This flow line segment is either part of a trapped flow line (one who spends infinite backwards or forwards time in a cusp) or it isn't.

Case 1: Suppose ℓ is not a subsegment of a trapped flow line. Extend ℓ a bit further if necessary so that the endpoints of ℓ lie outside the cusps; then ℓ and its associated pushed flow line $\overline{\ell}$ have the same endpoints. Let γ be the geodesic connecting these endpoints; then $\overline{\ell}$ lives inside the R_s-neighborhood of γ_+, the string of beads along γ by Proposition 3.1.3. The flow line ℓ agrees with $\overline{\ell}$ outside the cusps, thus $\ell \cap \overline{\ell}$ lies inside the R_s-neighborhood of γ_+ as well.

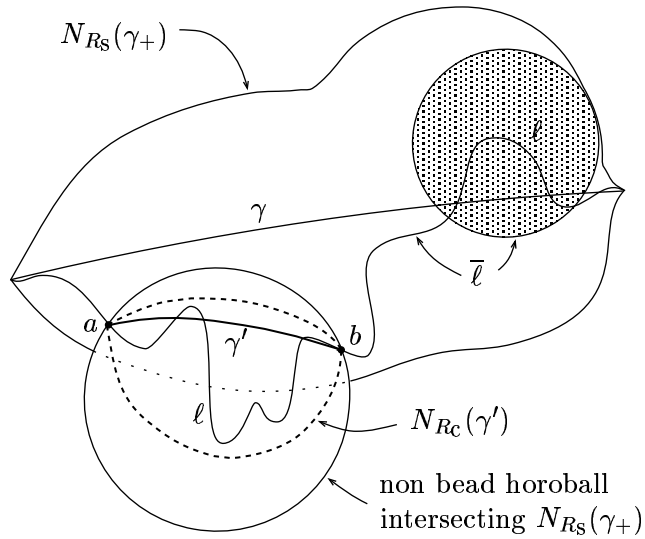

Figure 3.3: Flow line segments lie inside $R_f = R_s + R_c$ neighborhoods of strings of beads.

Consider a subsegment of ℓ which lies inside a cusp, with endpoints a and b on the boundary of the cusp (see Figure 3.3). Since the a and b are points on the pushed flow line, they lie within a distance R_s of the string of beads along the geodesic. In fact, since horoballs do not intersect, these points a and b must lie within radius R_s of the geodesic γ. Therefore (since geodesic segments are furthest apart at their endpoints), the entire geodesic segment γ' connecting these endpoints lies within a distance R_s of γ. But a flow line is quasigeodesic inside the cusps, therefore ℓ lies within a neighborhood of some radius R_c of γ'. Hence, the portions of flow line inside the cusp (and thus the entire flow line segment) lie within a neighborhood of radius $R_f = R_s + R_p$ of the string of beads along γ.

Case 2: Suppose now that ℓ is a subsegment of a trapped flow line. Extend this flow line segment all the way to its end, where the trapping horoball meets the sphere at infinity. Let $\overline{\ell}$ be the associated pushed flow line; then again $\overline{\ell}$ has the same endpoints as ℓ, with at least one of these endpoints lying on the sphere at infinity. Let γ be the geodesic connecting these endpoints. Then the trapping horoball(s) are beads on the the string of beads along γ, so it is immediate that the portion of ℓ inside the trapping horoball lies within the string of beads, and hence any neighborhood of the string of beads, along γ. The argument for the rest of ℓ follows exactly as above. Therefore, every flow line segment lies within a neighborhood $R_{\mathrm{f}} = R_{\mathrm{s}} + R_{\mathrm{c}}$ of the string of beads connecting its endpoints. q.e.d.

3.3 Flow Line Segments Lie Close to Geodesics

We now know that a flow line segment is close to a string of beads. The goal of this section is to show that a flow line segment is close to the *string* in the string of beads.

The proof proceeds as follows: Consider a flow line segment ℓ which lies in an R_{f} neighborhood of a bead. We will show that the subsegments of ℓ which lie near the bead but do not enter the bead have a bounded length M_{f}, where the bound depends only on R_{f}, and therefore these subsegments are close to the string. On the other hand, since flow lines are quasigeodesic inside cusps, the subsegments of ℓ which lie inside the bead lie within R_{c} of some geodesic segment in the bead. We will show this geodesic lies within an $R_{\mathrm{f}} + M_{\mathrm{f}}$ neighborhood of the string, and

therefore the subsegment of ℓ inside the bead lies within $R_{\mathrm{f}} + M_{\mathrm{f}} + R_{\mathrm{c}}$ of the string.

Lemma 6 (Limited Time Lemma) *For any $R > 0$, there is a maximum length M, depending only on R, of a segment of flow line which lies within hyperbolic distance R of a horoball without entering the horoball.*

Proof: The modified metric is just the singular Solv metric outside the cusps, cusps are vertical, star-shaped cylinders as shown in Figure 2.2, and flow lines are vertical straight lines given by a fixed choice of r and θ in this metric. Let C be a cusp neighborhood in the infinite cyclic cover of M, and let B be the region between a cusp and a (hyperbolic) R-neighborhood of the cusp, and consider the collection of flow lines which intersect B. Each flow line meets B in at most 2 intervals, whose total length is finite, for the shape of the cylinder will force each end of the flow line to either exit the R-neighborhood of the cusp or enter the cusp itself. Each flow line must have at least one endpoint intersecting the boundary of $N_R(C)$. For each flow line, there exists an isometry which takes the point of intersection with $N_R(C)$ at height z_h to height $z = 0$. Thus we have a compact segment's worth of flow lines, each of finite length, whose lengths vary continuously (see Figure 3.4). By compactness, then, there is a maximum such length for all flow lines with θ values in $[0, n\pi]$.

An explicit calculation, which we omit, shows that we may take the bound to be $M = 12e^B$, where B is the maximum hyperbolic length of a geodesic segment that can lie within an R neighbornood of a horoball without entering the horoball. q.e.d.

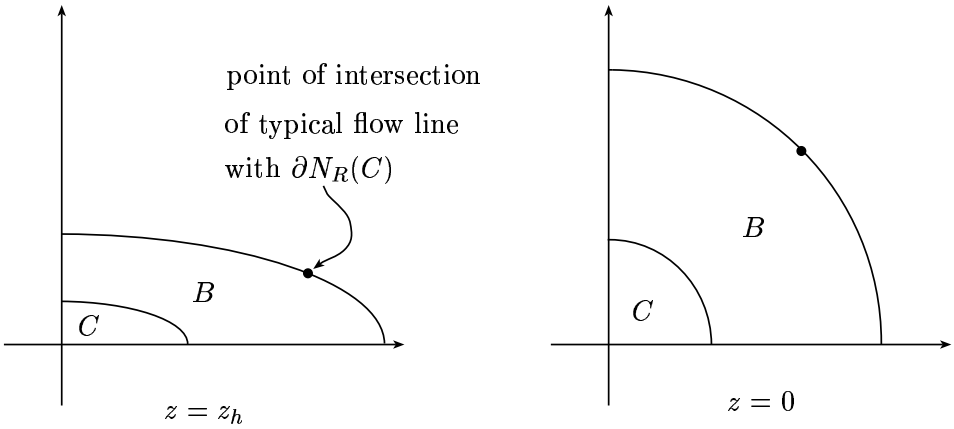

Figure 3.4: A cusp neighborhood C and the region B, before and after an isometry.

3.3.1 Flow Line Segments Lie Inside Neighborhoods

Theorem 3.3.1 *There is a constant $R_\Phi > 0$ such that every segment of flow line of the suspension flow on M lies within the neighborhood of radius R_Φ of the geodesic connecting its endpoints.*

Proof: Let ℓ be a flow line segment, γ be the geodesic segment with the same endpoints, and γ_+ be the string of beads along γ. Then ℓ lies within a neighborhood of radius R_f of γ_+, by Proposition 3.2.1. Away from the beads, ℓ lies within R_f of the string, so we only need to consider the portions of ℓ which lie within R_f of a bead. Let a and b be the points where ℓ last enters and first exits the R_f neighborhood of a bead H.

Note that a flow line must pass within R_f of the string when passing from an R_f neighborhood of one bead to an R_f neighborhood of another bead: even if R_f is so large that the R_f-neighborhood of one bead along

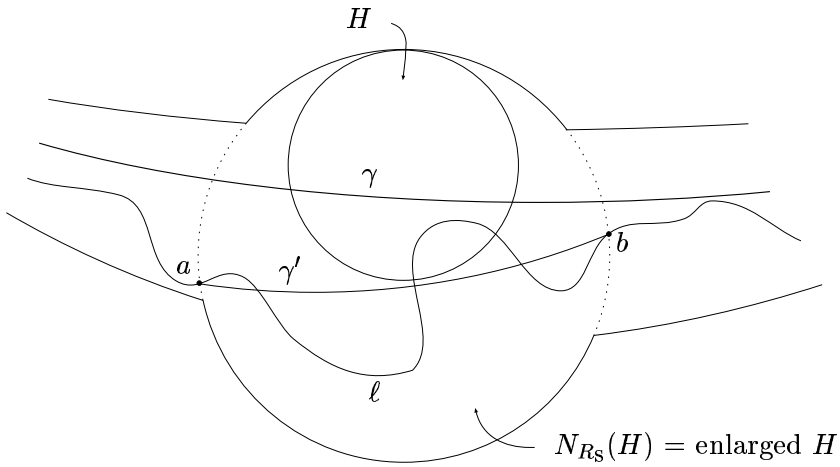

Figure 3.5: A flow line segment lies near the string in the string of beads.

the string of beads intersects the R_f neighborhood the next bead along the string of beads, the region intersection of these two bead neighborhoods will live inside the R_f neighborhood of the string. Therefore, a and b lie inside the R_f neighborhood of the string of beads.

Let M_f be the constant that comes from the Limited Time Lemma (Lemma 6) for $R = R_f$. This segment of ℓ between a and b is either shorter than or longer than M_f. If the segment is shorter than M_f, then the entire segment is within $R_s + M_f$ of the geodesic, by measuring along the segment back to endpoint a (at most M_f), then joining up to the string with a path of length at most R_s.

So assume that the segment is longer than M_f. Then by the Limited Time Lemma, the flow line segment must enter the cusp (see Figure 3.6) The flow line segment does not enter the cusp more than once, by Lemma 2. Therefore this flow line segment consists of a segment from a to the point a' where the flow line meets the cusp (of length at most M_f), connected to a segment from a' to b' which lies entirely within the cusp,

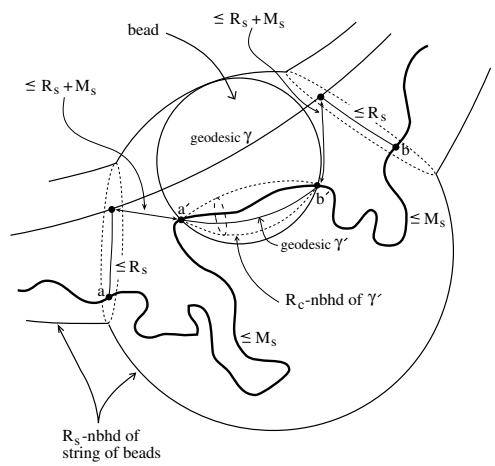

Figure 3.6: The flow line segment ℓ lies within $R_\Phi = R_\mathrm{c} + R_\mathrm{s} + M_\mathrm{f}$ of γ.

connected to a segment from b' to b of length at most M_f. As above in the case of the short flow line segment, the subsegments between a and a' and between b and b' are within $R_\mathrm{s} + M_\mathrm{f}$ of the geodesic, by measuring along the subsegments. It remains only to consider the portion of the $|fl$ inside the cusp.

The flow lines are quasigeodesic in the cusp, therefore the portion of ℓ inside H lies within a neighborhood of radius R_c of the geodesic connecting its endpoints a' and b'; call it γ'. The maximum distance between two geodesic segments occurs at one of the endpoints of the segments; thus the entire geodesic segment γ' lies within a $R_\mathrm{s} + M_\mathrm{f}$ neighborhood of γ. Therefore, every point on ℓ between a' and b' lies within a distance of $R_\mathrm{c} + R_\mathrm{s} + M_\mathrm{f}$ of γ. Taking $R_\Phi = R_\mathrm{c} + R_\mathrm{s} + M_\mathrm{f}$ gives the result. q.e.d.

Chapter 4

Flow Line Progress

In this section, we complete the proof by showing that flow lines satisfy part 2 of the definition of tracking: that inside the R_Φ neighborhood of the geodesic connecting its endpoints, a flow line makes progress from one end of the tube to the other.

Let ℓ be a flow line, γ be the geodesic connecting its endpoints, and let $\pi_\gamma : \mathbb{H}^3 \to \gamma$ denote nearest point retraction onto γ. Then $\ell \subset N_{R_\Phi}(\gamma)$. The set of all horoballs meeting $N_{R_\Phi}(\gamma)$ can be split into two types: **big horoballs**, which disconnect the neighborhood $N_{R_\Phi}(\gamma)$, and **small horoballs**, which do not.

The big horoballs split the neighborhood $N_{R_\Phi}(\gamma)$ up into segments. A flow line may enter a horoball in $\widetilde{M}$ only once. Therefore, since the flow line has two distinct endpoints, it must enter a big horoball from one segment of $N_{R_\Phi}(\gamma)$, and emerge from the big horoball to a different segment.

Lemma 7 *There is a constant B_1 so that the hyperbolic distance any flow line travels inside a small horoball is at most B_1.*

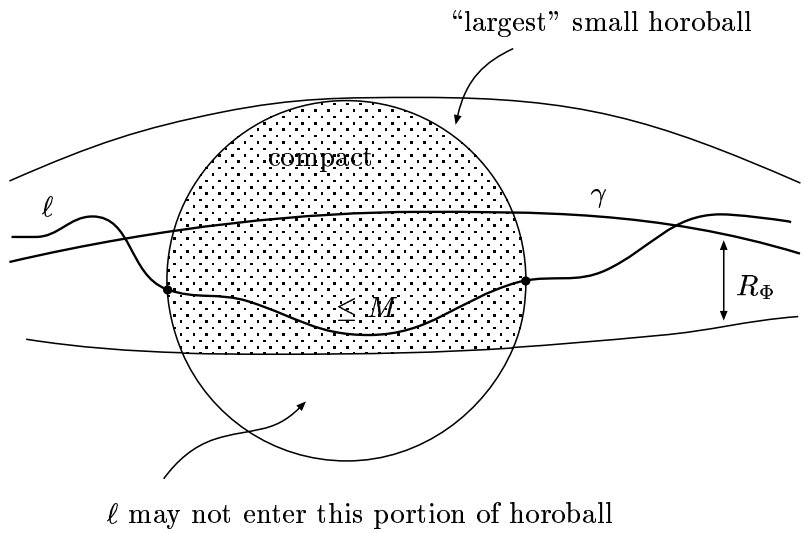

Figure 4.1: The maximum length flow segment inside a small horoball.

Proof: There is a maximum diameter to the intersection between a small horoball and the tube of radius R_Φ, beyond which the small horoball becomes a big horoball (see Figure 4.1). Therefore there is a maximum distance B_1 a flow line can travel inside a small horoball, given by the quasigeodesic constant K_c in the horoball times the maximum geodesic distance across $\{\text{small horoball}\} \cap N_{R_\Phi}(\gamma)$. q.e.d.

Theorem 4.0.2 *There exists $Q_\Phi > 0$ such that if ℓ is a segment of a flow line with endpoints a and b of length at least Q_Φ, then the distance between $\pi_\gamma(a)$ and $\pi_\gamma(b)$ along γ is at least 1.*

Proof: Let ℓ be a flow line and let γ be the geodesic connecting its endpoints, so that $\ell \subset N_{R_\Phi}(\gamma)$. Split this flow line into segments at the

boundaries of the big horoballs, so that some segments lie entirely inside a big horoball, and the rest of the segments lie between big horoballs.

First we will consider segments of the flow line which lie inside a big horoball. Let α be a segment of ℓ from the point at which ℓ enters a big horoball to the point it exits the big horoball. Since ℓ may enter this big horoball at most once, one endpoint of α must lie on one side of the big horoball (towards one endpoint of γ on the sphere at infinity), and the other endpoint of α must lie on the other side of the big horoball (see Figure 4.2).

Since α lies entirely inside a big horoball, α is quasigeodesic. Thus it tracks the geodesic segment γ' connecting its endpoints. Let R', Q' be the tracking constants given in the definition of tracking, and suppose a and b are two points on α. Since nearest point retraction onto each geodesic is distance non-increasing, we know

$$d(\pi_\gamma a, \pi_\gamma b) \geq d(\pi_\gamma(\pi_{\gamma'} a), \pi_\gamma(\pi_{\gamma'} b))).$$

Each endpoint of γ' lies within R_Φ of γ, thus the entire segment γ' lies within the bounded distance R_Φ of γ. Therefore, nearest point retraction of γ' onto γ will decrease distance, but only by a bounded amount. Thus there exists a $W \geq 1$ such that

$$d(\pi_\gamma(\pi_{\gamma'} a), \pi_\gamma(\pi_{\gamma'} b)) \geq \frac{1}{W} \cdot d(\pi_{\gamma'} a, \pi_{\gamma'} b).$$

Finally, we know that by the choice of Q',

$$d_\alpha(a, b) > Q' \;\Rightarrow\; d(\pi_{\gamma'} a, \pi_{\gamma'} b) > 1.$$

Putting the above equations together, if a and b are points on α for which $d_\alpha(a, b) > W \cdot Q'$, then $d(\pi_{\gamma'} a, \pi_{\gamma'} b) > W$ and therefore $d(\pi_\gamma a, \pi_\gamma b) > 1$.

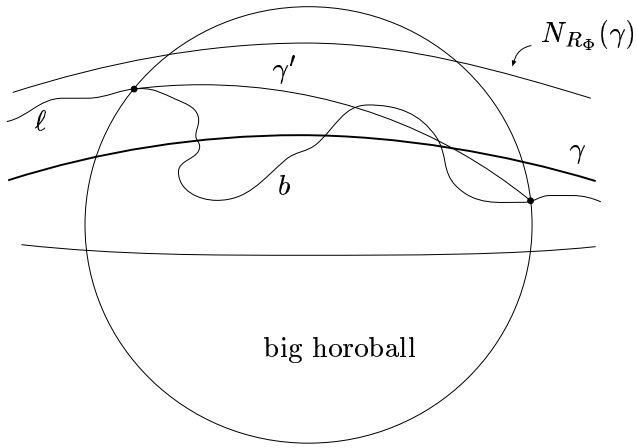

Figure 4.2: The segments of ℓ inside a big horoball track γ.

Thus all flow segments which lie inside a big horoball make progress along γ. The problem is now reduced to showing that flow lines make progress along the segments between the big horoballs. Let $|\ \ |_{\mathbb{H}^3}$ denote length in the hyperbolic metric, and $|\ \ |_m$ stand for length in the modified singular Solv metric on M.

Suppose a and b are the endpoints of a flow line segment β of hyperbolic length L which lies in a single segment of $N_{R_\Phi}(\gamma)$ between big horoballs. Consider a subsegment β' of β, obtained by moving from points a and b along β, if necessary, to the points a' and b' where the neutered space of M is first entered. Let L be the hyperbolic length of β'; then by Lemma 7, the hyperbolic length of β is at most $L + 2B$. The flow line segment β' may pass through many small horoballs. The horoballs in M were chosen to be a hyperbolic distance at least 1000 apart. Therefore, β' is made up of segments of hyperbolic length at least 1000 outside the cusps, alternating with segments of hyperbolic length at most B inside cusps (Lemma 7). So every segment of β' of hyperbolic

length $1000 + B$ must have a subsegment of hyperbolic length at least 1000 outside the cusps.

Let C be the number of segments of β' of hyperbolic length $1000+B$, so that

$$C \cdot (1000 + B) \leq |\beta'|_{\mathbb{H}^3} = L < (C + 1) \cdot (1000 + B).$$

Then β' has a collection S of subsegments in the neutered space of M of combined hyperbolic length at least $1000 \cdot C$. Let $|S|$ denote the combined length of the subsegments in S. Since the hyperbolic metric on M is K-quasi-isometric to the modified metric on M, we know that

$$|S|_m \geq \frac{1}{K} \cdot |S|_{\mathbb{H}^3} \geq \frac{1}{K} \cdot 1000C > \frac{1000}{K} \cdot \left(\frac{L}{1000 + B} - 1 \right).$$

Recall that flow lines are vertical straight lines in our model of the modified metric, so that in this model, a, a', b', and b are points with identical x and y coordinates, but different t coordinates. The collection S of subsegments of β' is in the neutered space N of M, where the modified metric is just the singular Solv metric. In singular Solv, the length of a vertical line segment is just the difference in the height coordinates of the endpoints. Therefore

$$\text{height difference between } a', b' \geq |S|_m \geq \frac{1000}{K} \cdot \left(\frac{L}{1000 + B} - 1 \right).$$

Let c be the unique path between a' and b' in $N_{R_\Phi}(\gamma)$ consisting of the shortest hyperbolic path (at most R_Φ) connecting a' to $\pi_\gamma(a')$ on γ, followed by a path of hyperbolic length D along γ, followed by the shortest hyperbolic path (again of hyperbolic length at most R_Φ) connecting γ at $\pi_\gamma(b')$ to b' (see Figure 4.3). Then the hyperbolic length of c is at most $2R_\Phi + D$. This path c need not be in the neutered space of

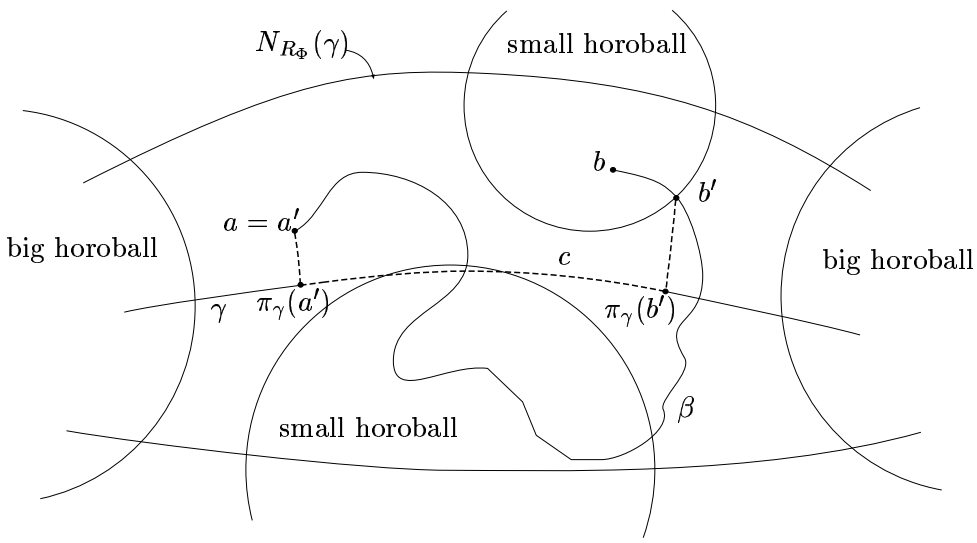

Figure 4.3: The path c between a' and b'.

M. However, c may not go into any small horoball a hyperbolic distance greater than B_1 by Lemma 7. Further, note that either of $\pi_\gamma a'$ or $\pi_\gamma(b')$ may lie in a big horoball, but both must be within a hyperbolic distance R_Φ of $N_{R_\Phi}(\gamma)$, and therefore every point of c inside a big horoball must be within hyperbolic distance $B_2 = 2R_\Phi + 1$ from $N_{R_\Phi}(\gamma)$ by the triangle inequality (Figure 4.4 illustrates this worst case scenario). Therefore all points of c lie within $B = \max(B_1, B_2)$ of $N_{R_\Phi}(\gamma)$.

Push c out of all horoballs into the neutered space to a path $\overline{c}$; this will increase its hyperbolic length by a factor of at most e^B, so that $|\overline{c}|_{\mathbb{H}^3} \leq e^B \cdot (2R_\Phi + D)$. Therefore, since the modified metric is K-quasi-isometric to the hyperbolic metric,

$$|\overline{c}|_m \leq e^B \cdot (2R_\Phi + D) \cdot K.$$

Now $\overline{c}$ is a path in the neutered space between a' and b', therefore the length of $\overline{c}$ in the modified metric is the same as the length of $\overline{c}$

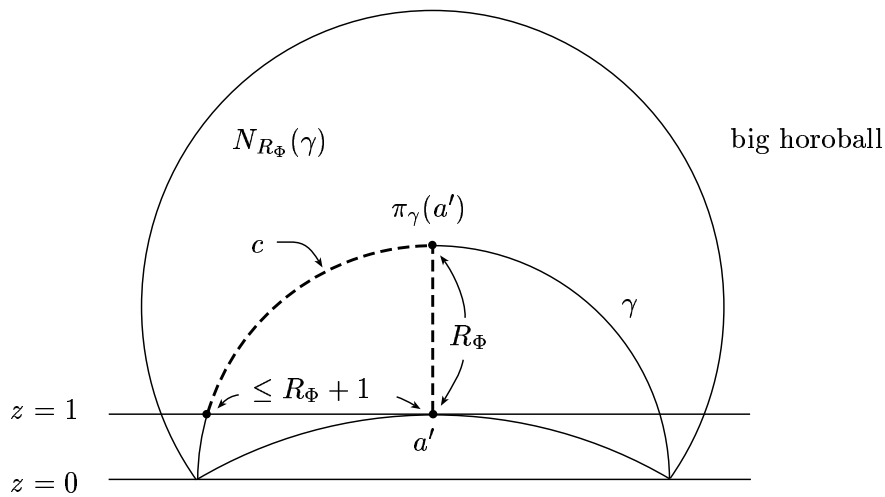

Figure 4.4: All points on c are within $2R_\Phi + 1$ of $N_{R_\Phi}(\gamma)$.

in the singular Solv metric. Recall that a' and b' are two points on a segment of a flow line β', and that flow lines are true geodesics in the singular Solv metric on M. Therefore, the height difference between the endpoints a' and b' is a lower bound for the singular Solv length of any path in the neutered space between a' and b'. Further, since the modified metric on the neutered space N of M is just the singular Solv metric on N, the height difference between the endpoints a' and b' is a lower bound for the length in the modified metric of any path in the neutered space between a' and b'. Therefore,

$$\begin{aligned} e^B \cdot (2R_\Phi + D) \cdot K &\geq |\overline{c}|_m \\ &\geq \text{height difference in endpoints } a',\, b' \\ &\geq \frac{1000}{K} \cdot \left(\frac{L}{1000 + B} - 1 \right), \end{aligned}$$

from which it follows that

$$D \geq \frac{1000}{K^2 e^B} \cdot \left(\frac{L}{1000 + B} - 1 \right) - 2R_\Phi.$$

The term on the right hand side of the inequality is linear in L. Therefore, there is a Q so large that if $L \geq Q-2B$, then $D > 1+2B$. So, if the hyperbolic length of β is at least Q, then the hyperbolic length of β' is at least $L \geq Q-2B$, from which it follows that $D = d_{\mathbb{H}^3}(\pi_\gamma(a'), \pi_\gamma(b') > 1 + 2B$, and therefore $d_{\mathbb{H}^3}(\pi_\gamma(a), \pi_\gamma(b)) > 1$.

Let $Q_\Phi = 3 \cdot \max(W \cdot Q', Q)$, and suppose ℓ is a segment of flow line with endpoints a and b of hyperbolic length at least Q_Φ. If ℓ enters two big horoballs, then as the horoballs separate $N_{R_\Phi}(\gamma)$ and are at least 1000 units apart, it follows that $d_{\mathbb{H}^3}(\pi_\gamma(a), \pi_\gamma(b)) > 1000$. If ℓ enters at most one big horoball H, then ℓ may be broken up into at most 3 segments by H: one segment inside H, and one segment on either side of H in $N_{R_\Phi}(\gamma) \setminus H$. Since $|\ell|_{\mathbb{H}^3} \geq Q_\Phi = 3 \cdot \max(W \cdot Q', Q)$, at least one of those segments must be of length at least $\max(W \cdot Q', Q)$, and therefore $d_{\mathbb{H}^3}(\pi_\gamma(a), \pi_\gamma(b)) > 1$, as desired. q.e.d.

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