Project 2 – Calculator Puzzler – Honors Calculus 2

If you like, you can work with one other student (in this class) on this project. You should write up your findings and the answers to all the questions below in the form of a report: use words and paragraphs, inserting the mathematics supporting the claims you make at the appropriate places in your report. As always, Written Homework Guidelines apply. Do not simply list the answers to the questions! Embed your answers in to a report / research paper which addresses this project as a whole.

Let's investigate how your calculator's $\sqrt{\ }$ (square-root) key works. You might wonder how you would compute a square-root yourself, or at least how you would approximate it. If you answered "I would use my calculator," remember that someone had to program your calculator to approximate the square root of a number, so that doesn't really answer the question of how one might carry out such an approximation.

Notice what happens when you repeatedly press the $\sqrt{\ }$ key on your calculator. Let's try this with 5 as the initial input number and see what sequence of numbers we get:

5
2.23606...
1.49534...
1.**222**84...
1.**105**82...
1.**051**58...
1.**025**46...
. . . . . .

It looks like the fractional part is roughly halved at each successive stage, that is, each time we take the square-root, the next number we get seems to have a fractional part about half that of the preceding number. But it can't be that that is all our calculator really does! What is going on?

Part A: Let's investigate...

- Does this halving process take place if we start with a number other than 5? Do you think this happens for all? most? some? choices for initial value? Experiment.
- When does this "halving" phenomenon happen in general? How big or small does the input number have to be for the (approximate) halving to occur starting at the first step?
- Did you investigate also what happens when your input number is a fraction?
- Write a coherent sentence / paragraph describing this phenomenon, and on what interval of real numbers you feel this phenomenon occurs.

Follow the steps below, assuming you are dealing with input numbers of the sort which you have experimentally determined would produce the "roughly half the fractional part" phenomenon.

Part B: Non-calculus Method

1. Let s be a "small", positive number, where you define the word "small" in terms of your investigation in part A. Then our observation that "the fractional part is roughly halved at each stage" can be mathematically described by: The square-root of $1 + s$ is approximated by $\sqrt{1+x} \approx$ __________
2. Give an informal proof/justification for this approximate equality by analyzing what you get by squaring both sides.

3. If $x = 1 + s$ then another way to describe our observations mathematically is: The square-root of x is approximated by $\sqrt{x} \approx$ ___________ (You should fill in the blank with an expression in terms of x)

4. Give an informal proof/justification of the above inequality.

Part C: Calculus to the Rescue!

1. Use the Taylor series of $f(x) = \sqrt{x}$ at $a = 1$ to give a more formal justification of your approximation in part B3. You may assume that $f(x)$is equal to its Taylor series, $T(x)$, so that you need only show that $T(x)$ is approximated by the right-hand side of the approximation B3.

2. Now use Taylor's Theorem with Remainder to quantify how good an approximation the approximate-equality in B3 is. For example, if x is within 1/100 of 1, how good an approximation of $\sqrt{x}$ is the right-hand side of B3?

3. What is the range of x values so that the approximation in B3 is accurate to within 1/1000?

Calculator Project Grading Rubric

Your report will be graded using the following rubric:

_____ / 2 pts **The solution is written as integrated report**, and not simply as list of answers to the questions. This report is made up of paragraphs, with the mathematical calculations and the data brought in to the report at the points where they are needed (with the exception that large tables of data may be placed in an appendix, and referred to at the appropriate place in the report). It is clear what the relationship between all the individual questions are, by the way the report is written. Actual calculations are given their own separate line or lines in the appropriate places in the report, and are seen as parts of sentences, rather than just appearing out of nowhere.

_____ / 3 pts **The mathematics is correct**

_____ / 3 pts **The reasoning behind the mathematics is shown clearly and correctly.** For example,

- All variables used in the project must be defined before their use.
- Any series written must be accompanied by its interval of convergence.
- Explanations / justification is given for all the steps in the report.

_____ / 2 pts **Report written in LaTeX**