Challenge Problems – Exam 1

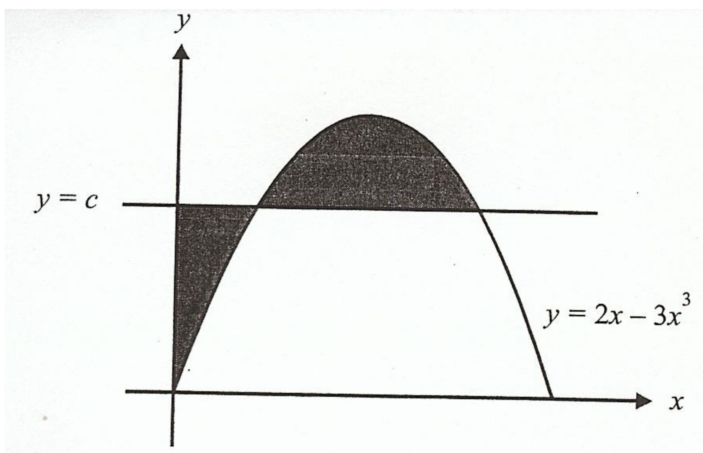
1. (a) State the Mean Value Theorem, making sure you clearly state the hypotheses and the conclusion.
   (b) Show that if $F(x)$ and $G(x)$ are antiderivatives of $f(x)$, then $F(x) - G(x) = C$, where C is a constant (does not depend on x). [Hint: Use the MVT on $H(x) = F(x) - G(x)$.]

2. Suppose 2 variables x and y are related such that $\sqrt{1-x^2}\,dy = x^3\,dx$. If $y = 0$ when $x = 0$, express y as a function of x.

3. If f is continuous and has nonnegative values, and if $f''(x) > 0$ throughout $[a, b]$, prove that $\int_a^b f(x)\,dx$ is less than the number given by the Trapezoid Rule.

4. Find all real-valued continuously differentiable functions f on the real line such that $\forall x$
$$[f(x)]^2 = \int_0^x (f(t))^2 + (f'(t))^2\,dt + 1990.$$

5. The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find c so that the areas of the two shaded regions are equal.



6. Show that $\int_0^\infty \sin(x)\sin(x^2)\,dx$ converges.

7. Consider numerically approximating the integral $\int_a^b f(x)\,dx$. Show that $T_{2n} = \dfrac{T_n + M_n}{2}$. That is, by averaging the trapezoid and the midpoint methods, you get exactly the same thing as if you did the trapezoid method with twice as many intervals. [Hint: Remember that the number of intervals in T_n, M_n, and T_{2n} are different, hence δx is different for all three.]

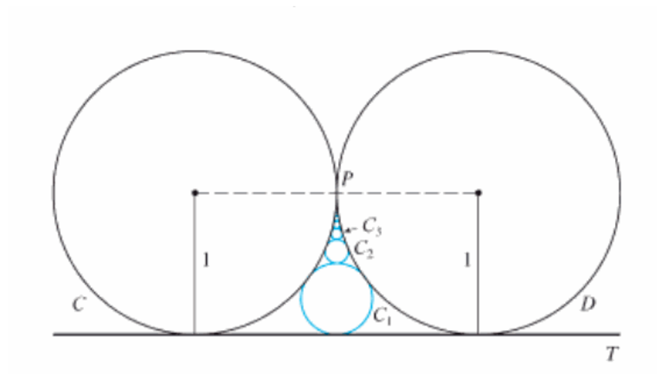
Challenge Problems – Exam 2

1. If $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} b_n$ is divergent, show that $\sum_{n=1}^{\infty} a_n + b_n$ is divergent. [Hint: suppose it is not true, and see if you can come to a contradiction.]

2. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both divergent, show by giving two particular examples that $\sum_{n=1}^{\infty} a_n + b_n$ could be either convergent or divergent.

3. If $\sum_{n=1}^{\infty} a_n$ diverges and c is a constant, prove that $\sum_{n=1}^{\infty} c \cdot a_n$ diverges.

4. Two circles C and D of radius 1 touch at P. T is a common tangent line; C_1 is the circle that touches C, D, and T; C_2 is the circle that touches C, D, and C_1; C_3 is the circle that touches C, D, and C_2, etc. Continue this procedure indefinitely to produce an infinite sequence of circles $\{C_n\}$. Find an expression for the diameter of C_n. Which in-class example is this a geometric representation of?



5. What is wrong with the following calculation?

$$\begin{aligned}
0 &= 0+0+0+\cdots \\
&= (1-1)+(1-1)+(1-1)+\cdots \\
&= 1-1+1-1+1-1+\cdots \\
&= 1+(-1+1)+(-1+1)+(-1+1)+\cdots \\
&= 1+0+0+0+\cdots \\
&= 1
\end{aligned}$$

6. (a) Use a graph of $y = \frac{1}{x}$ to show that if s_n is the nth partial sum of the harmonic series, then $s_n \leq 1 + \ln n$.
   (b) The harmonic series diverges, but very slowly. Use part (a) to show that the sum of the first million terms is less than 15, and the sum of the first billion terms is less than 22.

7. Is there an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that $a_1^m + a_2^m + a_2^m + \cdots = m$ for every positive integer m?

8. For which positive integers k is the following series convergent? $\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$

9. Around 1910, Srinivasa Ramanujan discovered the formula

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}.$$

   William Gosper used this series in 1985 to compute the first 17 million digits of π.

   (a) Verify that the series is convergent
   (b) How many correct decimal places of π do you get if you use just the first term of the series? What if you use the first 2 terms?

Challenge Problems – Exam 3

1. Use the power series for $\arctan(x)$ to prove the following expression for π:

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

2. Estimate $\cos(69°)$ correct to 5 decimal places.

3. (a) Show that the Bessel function of order 0 (defined in class) satisfies the differential equation

$$x^2 J_0''(x) + xJ_0'(x) + x^2 J_0(x) = 0.$$

   (b) Evaluate $\int_0^1 J_0(x)$ accurate to 3 decimal places.

4. A car is moving with speed 20 m/s and acceleration 2m/s^2 at a given instant. Using a second degree Taylor polynomial, estimate how far the car moves in the next second.

5. If a surveyor measures differences in elevation when making plans for a highway across a desert, corrections must be made for the curvature of the Earth.

   (a) If R is the radius of the Earth and L is the length of the highway, show that the correction is

$$C = R\sec(L/R) - R.$$

   (b) Use a Taylor polynomial to show that

$$C \approx \frac{L^2}{2R} + \frac{5L^4}{24R^3}.$$

   (c) Compare the corrections given by the formulas i n parts (a) and (b) for a highway that is 100km long. Take the radius of the Earth to be 6370 km.

6. Let $A(t)$ be the area of a tissue culture at time t and let M be the final area of the tissue when growth is complete. Most cell divisions occur on the periphery of the tissue and the number of cells on the periphery is proportional to $\sqrt{A(t)}$. So a reasonable model for growth of tissue is obtained by assuming that the rate of growth of the area is jointly proportion to $\sqrt{A(t)}$ and $M - A(t)$. Formulate a differential equation and use it to show that the tissue grows fastest when $A(t) = \frac{1}{3}M$.