Some Formulas and Notation – Complex Analysis

- Let $\alpha : [a,b] \to \Gamma$ be a parametrization of the curve Γ in $\mathbb{R}^2$. Then
  - α has two coordinate functions: $\alpha(t) = (\alpha_1(t), \alpha_2(t))$
  - tangent vector: $T(t) = (\alpha_1'(t), \alpha_2'(t))$
  - outward normal vector: $N(t) = (\alpha_2'(t), -\alpha_1'(t))$
  - length$(\alpha) = \displaystyle\int_a^b |\alpha'(t)| dt$

- Local Linearity: Let $u \in \mathscr{C}^1(\Omega)$ with $z_0 \in \Omega$. For each $z \in \Omega$ we can write

$$u(z) = u(z_0) \ + \ u_x(z_0) \cdot (x - x_0) \ + \ u_y(z_0) \cdot (y - y_0) \ + \ \text{ higher order terms,}$$

  where the higher order terms $\to 0$ as $z \to z_0$

- gradient of $u = \nabla u = (u_x, u_y)$.

- The directional derivative of u at a point P in the direction of $\vec{v}$ is: $D_{\vec{v}} u(P) = \nabla u(P) \cdot \frac{\vec{v}}{|\vec{v}|}$.
  It measures the degree to which ∇u lines up with $\vec{v}$ at P.

- Outward Normal Derivative at z_0: $\dfrac{\partial u}{\partial n}(z_0) = \nabla u(z_0) \cdot N(z_0)$ [N must be unit vector $\perp$ to α]

- radius function $r(z) = |z| = \sqrt{x^2 + y^2}$

- argument function $\arg(z) = \theta(z) = \arctan(y/x)$, $-\pi < \theta \leq \pi$.

- Equality of Mixed partials: If $u_{xy}(z)$ and u_{yx} exist and are continuous, then they are equal.

- Laplacian of $u = \Delta u = u_{xx} + u_{yy}$.

- u is *harmonic in* Ω iff $\Delta u(z) = u_{xx}(z) + u_{yy}(z) = 0$ for all $z \in \Omega$.

- Line integral measures the degree to which the vector field $\big(p(x,y),\ q(x,y)\big)$ lines up with the *tangent* vectors to the curve Γ, accumulated across every point of Γ:

$$\int_\Gamma p\ dx + q\ dy \ = \ \int_\Gamma (p,q) \cdot T \ = \ \int_a^b \big(\ p(\alpha(t)),\ q(\alpha(t))\ \big) \ \cdot \ \big(\alpha_1'(t),\ \alpha_2'(t)\big)\ dt$$

- $du = u_x\ dx + u_y\ dy$

- Thm: If Γ is curve from z_0 to z_1, and $u \in \mathscr{C}^1(\Omega)$, then $\displaystyle\int_\Gamma du = u(z_1) - u(z_0)$

- Green's Theorem: If $p(x,y), q(x,y) \in \mathscr{C}^1(\Omega^+)$, then $\displaystyle\int_{\partial\Omega} [p\ dx + q\ dy] = \iint_\Omega (q_x - p_y)\ dx\ dy$

- Fund Thm of Calc: $\displaystyle\int_{\partial\Omega}$ function $= \displaystyle\iint_\Omega$ derivative of function

- net flux across $\partial\Omega$ (*normal* direction) is

$$\int_{\partial\Omega} \frac{\partial u}{\partial n}\, ds \;=\; \int_{\partial\Omega} \nabla u \cdot N \;=\; \int_a^b \big(\, u_x(\alpha(t)),\ u_y(\alpha(t))\,\big) \cdot \big(\alpha_2'(t),\ -\alpha_1'(t)\big)\ dt \;=\; \int_{\partial\Omega} [u_x\ dy - u_y\ dx]$$

- Inside-Out Theorem: If $u \in \mathscr{C}^2(\Omega^+)$, then $\displaystyle\int_{\partial\Omega} \frac{\partial u}{\partial n}\ ds = \iint_\Omega \Delta u\ dx\ dy$
  left hand side = "net change of u across boundary = net flux across boundary". The theorem says this net change can be obtained by integrating Δu across inside Ω.

- Thm 1: Let u be a harmonic function in $\mathscr{C}^2(\Omega^+)$. Then

  1. net flux across entire boundary is zero : $\displaystyle\int_{\partial\Omega} \frac{\partial u}{\partial n}\ ds = 0$
  2. a harmonic function is determined by its boundary behaviour:
     for any $z_0 \in \Omega$, we can find the value of $u(z_0)$ by computing the following integral involving only points z on $\partial\Omega$:

$$u(z_0) = \frac{1}{2\pi} \int_{\partial\Omega} \left(\ln r \frac{\partial u}{\partial n}(z) - u(z) \frac{\partial \ln r}{\partial n} \right)\ ds \qquad (\text{where } r = |z - z_0|)$$

- Characterization of Harmonic Functions:
  $u \in \mathscr{C}^2(\Omega)$ is harmonic iff for every Jordan curve Γ whose interior lies in Ω, we have $\displaystyle\int_{\partial\Omega} \frac{\partial u}{\partial n}\ ds = 0$

- Bump Principle: If q is cts on Ω, $q(z) \geq 0$ all $z \in \Omega$, then $q \equiv 0$ (zero function) iff $\iint_\Omega q\ dx\ dy = 0$