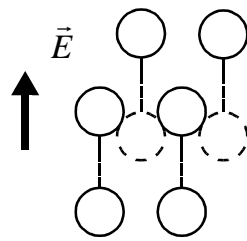# Dielectric Materials and Polarization

Dielectric Materials and Polarization

In order to understand the relationship between electric field $\vec{E}$ and stored energy (and power flux for travelling waves, $E^2/2\eta$), we need to take a microscopic look at a dielectric material to justify the relation $\vec{D} = \varepsilon_o \varepsilon_r \vec{E}$. After we establish this relationship, we must consider energy storage in a dielectric.

All matter is composed of atoms in some physical arrangement in space with other atoms. An atom is composed of a positively charged nucleus, surrounded by one or more electrons. The composite bulk material may be arranged in such a way that the electrons are free, or nearly free, to move about within the material. Metals and doped semiconductors are examples. Many more solids are insulators. In an insulator, all of the electrons are *bound* in atomic ore molecular orbitals and are not free to move about the material. In the case of a metal, an applied electric field will induce a current composed of the freely mobile electrons. In an insulator, or dielectric, something else must happen because there are no free electrons.

What can, and does, happen is that the electric field pulls the electron and pushes the positively charged nucleus. The result is a distorted object with a net atomic dipole moment, or polarization. This is electronic polarization. There is also orientational polarization (water) and ionic polarization (salt crystals).

This microscopic polarization, $\vec{p}$, of an individual atom, molecule, unit cell, etc. must be summed up to form the *macroscopic polarization $\vec{P}$*.

Formally:

$$\vec{P} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i \to 1}^{n\Delta v} \vec{p}_i = n\vec{p}_a = nQd_a = \rho_+ \vec{d}_a \,,$$

    where $n\Delta v$ = number of dipoles in a volume Δv.

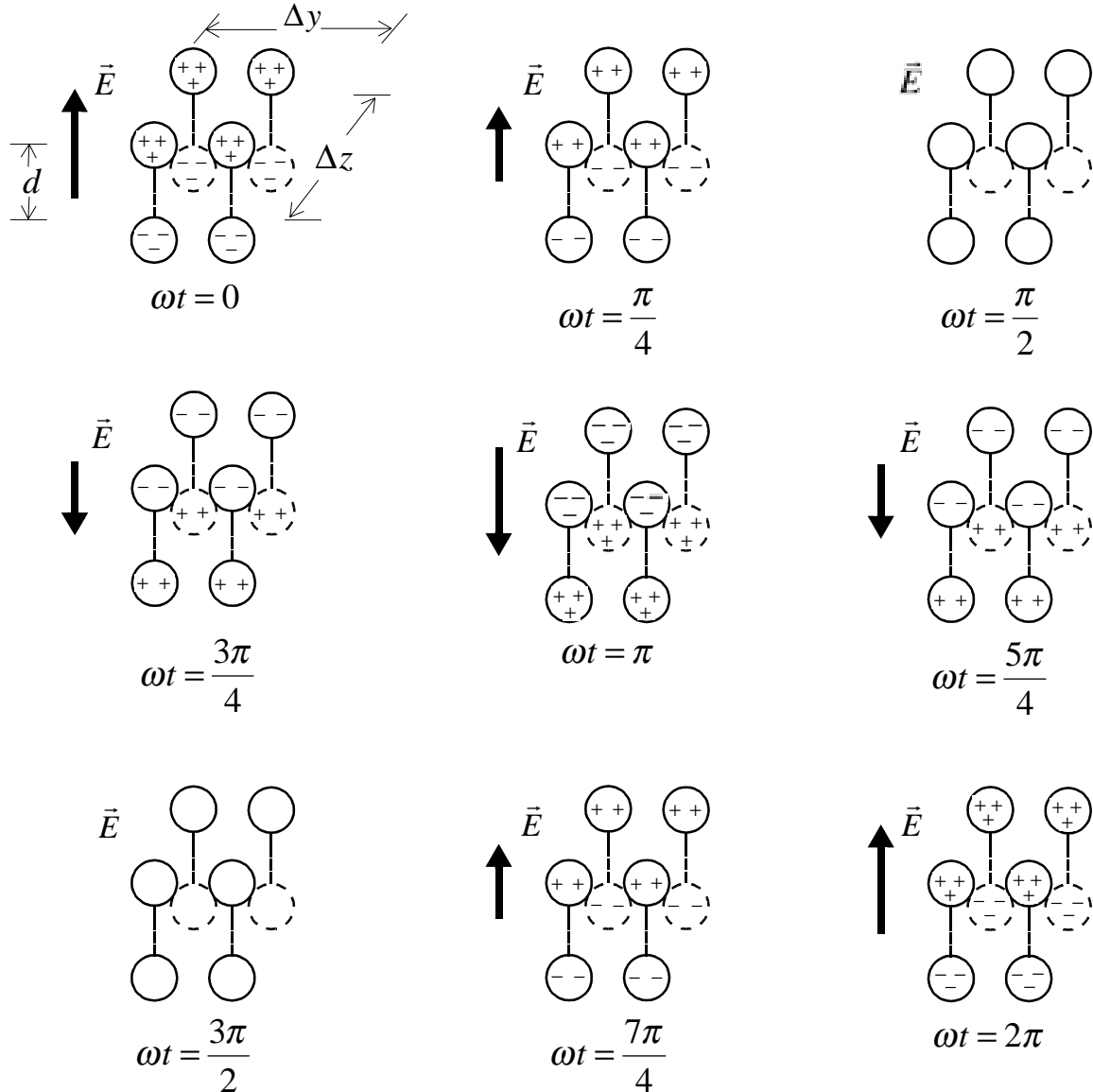$\vec{p}_a$ is the average dipole moment per atom or molecule.

$\vec{d}_a$ is the average vector separation distance between positive and negative charges.

$\rho_+ = nQ$, the density of positive charge created in the polarized region.

Note that, in the absence of an applied field, there is no polarization.

Polarization Current

Supppose we apply a time varying field to a polarizable medium (dielectric), where $\vec{E} = E_o \cos\omega t\, \hat{e}_x$. The result of this field is graphically depicted below:

$$\omega t = 0 \qquad \omega t = \frac{\pi}{4} \qquad \omega t = \frac{\pi}{2}$$

$$\omega t = \frac{3\pi}{4} \qquad \omega t = \pi \qquad \omega t = \frac{5\pi}{4}$$

$$\omega t = \frac{3\pi}{2} \qquad \omega t = \frac{7\pi}{4} \qquad \omega t = 2\pi$$

The flow of charge across the area $\Delta y\,\Delta z$ is equivalent to an induced oscillating current, called the polarization current. We need to quantify this polarization current. Let us assume a linear relationship, where the field $\vec{P} = \varepsilon_o\, \chi_e\, \vec{E}$; χ_e is called the electric susceptibility. It merely quantifies how easy it is for the field to produce a polarization in the particular material. We have $\vec{P} = \varepsilon_o\, \chi_e\, E_o \cos\omega t\, \hat{e}_x$.

The polarization is defined by the dipole moment per unit volume, so the total dipole moment in the volume $d\,\Delta y\,\Delta z$ is:

$$\vec{P}\,\Delta v = d\,\Delta y\,\Delta z\,\varepsilon_o\,\chi_e\,E_o\cos\omega t\,\hat{e}_x$$

This is equivalent to two charges, each equal to $Q = \varepsilon_o\,\chi_e\,\Delta y\,\Delta z\,E_o\cos\omega t$, separated by distance d. The associated current is:

$$\vec{J} = \frac{I}{\Delta y\,\Delta z} = \frac{1}{\Delta y\,\Delta z}\frac{dQ}{dt}\,\hat{e}_x = \left(-\,\omega\,\varepsilon_o\,\chi_e\sin\omega t\right)E_o$$

Recalling that $\vec{P} = \varepsilon_o\,\chi_e\,E_o\cos\omega t\,\hat{e}_x$, we can identify:

$$\boxed{\vec{J}_p = \frac{\partial\vec{P}}{\partial t} = \frac{\partial\left(\varepsilon_o\chi_e\vec{E}\right)}{\partial t}}$$

The induced polarization current is equal to the time rate of change of the polarization.

Ampere's Law is: $\dfrac{\nabla\times\vec{B}}{\mu_o} = \vec{J} + \dfrac{\partial\left(\varepsilon_o\vec{E}\right)}{\partial t}$

But now we need to add $\vec{J}_p$, the polarization current.

$$\frac{\nabla\times\vec{B}}{\mu_o} = \vec{J} + \frac{\partial\left(\varepsilon_o\vec{E}\right)}{\partial t} + \frac{\partial\vec{P}}{\partial t}$$

$$= \vec{J} + \frac{\partial\left(\varepsilon_o\vec{E}\right)}{\partial t} + \frac{\partial\left(\varepsilon_o\chi_e\vec{E}\right)}{\partial t}$$

$$= \vec{J} + \frac{\partial}{\partial t}\left[\varepsilon_o\left(1+\chi_e\right)\vec{E}\right]$$

$$\varepsilon_r = 1 + \chi_e$$

Then Ampere's Law is (in a dielectric):

$$\frac{\nabla\times\vec{B}}{\mu_o} = \vec{J} + \frac{\partial\left(\varepsilon_o\varepsilon_r\vec{E}\right)}{\partial t}$$

$$= \vec{J} + \frac{\partial\left(\varepsilon\vec{E}\right)}{\partial t}$$

$$= \vec{J} + \frac{\partial\vec{D}}{\partial t}, \text{ where } \vec{D} = \varepsilon\vec{E}$$

Polarization Charge Density

$\nabla \cdot \vec{P} = -\rho_p$ polarization charge density. More on this later.

Gauss' Law for Electric Fields in Materials

We have this new charge density that we must include:

$$\nabla \cdot \varepsilon_o \vec{E} = \rho_v + \rho_p \qquad \rho_v = \text{external } \textit{free} \text{ charge}$$

$$\rho_p = \text{induced charge in the dielectric}$$
$$\text{(This is a "bound" charge. It}$$
$$\text{disappears if } \vec{E} \text{ is removed.)}$$

$$\nabla \cdot \varepsilon_o \vec{E} = \rho_v - \nabla \cdot \vec{P} \, ,$$

$$\text{or} \quad \nabla \cdot \left(\varepsilon_o \vec{E} + \vec{P} \right) = \rho_v$$
$$\uparrow$$
external, "free" charge only (or "placed")

Now we make the linearity assumption: $\vec{P} = \chi \varepsilon_o \vec{E}$

Then:

$$\nabla \cdot \left(\varepsilon_o \vec{E} + \vec{P} \right) = \nabla \cdot \left(\varepsilon_o \vec{E} + \chi \varepsilon_o \vec{E} \right) = \rho_v \, ,$$

$$\text{or} \quad \nabla \cdot \varepsilon_o \vec{E}(1 + \chi) = \rho_v$$

But $\vec{D} = \varepsilon_o \varepsilon_r \vec{E} = \varepsilon_o E(1 + \chi)$,

so that:

$$\boxed{\nabla \cdot \vec{D} = \rho_v}, \quad \text{where} \quad \vec{D} = \varepsilon_o \varepsilon_r \vec{E}$$

$$\varepsilon_r = 1 + \chi_e$$

Note that $\nabla \cdot \left(\varepsilon_o \vec{E} + \vec{P} \right) = \rho_v$ is the most fundamental form of this equation which holds, even when $\vec{P}$ is not linearly proportional to $\vec{E}$. If $\vec{P}$ has terms proportional to 2nd, 3rd, etc. powers of $\vec{E}$, we have **nonlinear** material **not** covered in this course. **But** lots of important devices are based on it; for example, nonlinear optics, modulation of lasers (which are important in optical [fiber optics] communication).

Dielectric Materials
(alternate derivation of $\nabla \cdot \vec{P} = -\rho_p$)

Consider a region of space, filled with some linear and isotropic material. If, in that region of space, there exists a constant E field $\vec{E} = E_o \hat{e}_x$, then there is polarization $\vec{P} = \hat{e}_x \varepsilon_o \chi_e E$, also uniform. Since the polarization is uniform, the number of atomic dipoles per unit volume is constant and, as can be seen in the sketch, the effects of all the charges cancel out (except at boundary, which we examine separately).
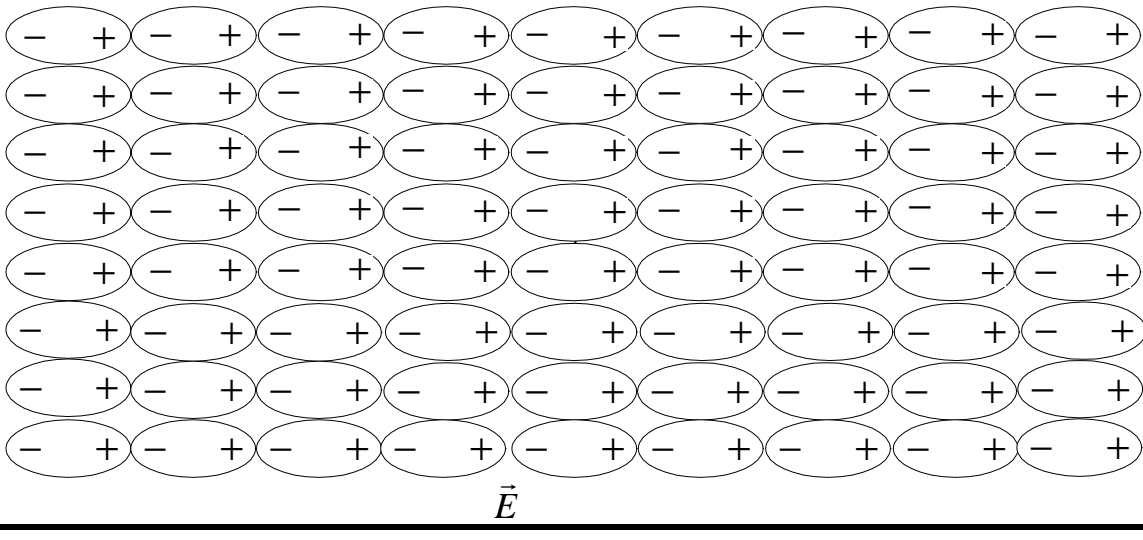
Now, consider that the E field is not constant, but increases linearly with x; that is, $\vec{E} = A x \hat{e}_x$. Then the polarization also increases linearly with x. That is to say, the number of aligned dipoles per unit volume increases linearly with x. Something like this is shown on the sketch.
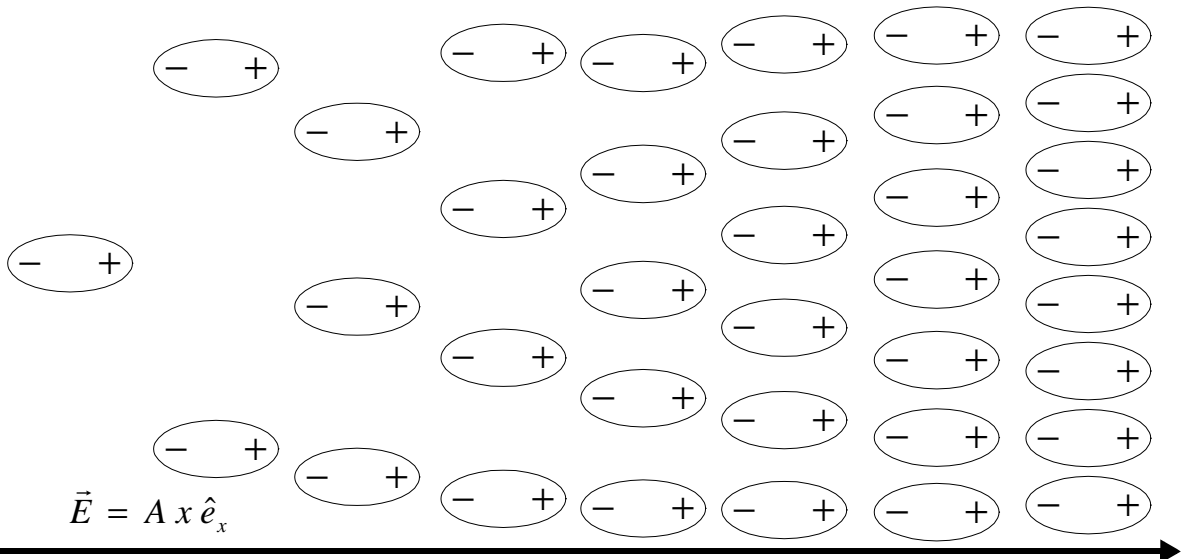
Obviously, now the plus and minus charges don't cancel out. There is always more negative charge to the right, so the effect is a *net negative charge in any given volume*. If $P = \hat{e}_x \, n(x) \, p_a$, where p_a is the atomic dipole and $n(x)$ is the number per unit volume (which is now **not** constant, but a function of x), then:

$$-\frac{dn}{dx} p_a = -\frac{dP}{dx} = \rho_v^b$$

Note that this charge density was not placed in as an excess charge, but comes from the non-uniform aligned dipoles. These charges are part of the material. If the field is removed, the dipoles randomize in orientation or dis-appear. So we distinguish these charges as a "bound" charge density.

In a more general form, we can write bound charge density due to dipoles as:

$$\rho_v^b = -\nabla \cdot \vec{P}$$

Any discontinuity at boundaries is also taken care of by the divergence of the polarization.

$$\vec{n} \cdot \left(P_1 - P_2 \right) = -\rho_s^b \text{ ***}$$

If $\vec{P}$ is uniform, but discontinuous at a boundary, $-\nabla \cdot \vec{P} \Rightarrow \vec{n} \cdot \vec{P}$, where $\vec{n}$ is outward normal and $\rho_v^b \Rightarrow \rho_s^b$

Thus, $\rho_s^b = \vec{n} \cdot \vec{P}$. (This is a special case of above.) This is clearly seen on the first sketch, which shows that, although all the charges cancel out for uniform $\vec{P}$, at the boundary $x = 0$, there is an effective **negative** surface charge density; and, at $x = a$, there is an effective **positive** surface charge density. This agrees with $\rho_s^b = \vec{n} \cdot \vec{P}$. This is the physical meaning of $\nabla \cdot \vec{P} = -\rho_s$. ***

Note that $\rho_v^b = -\nabla \cdot \vec{P}$ is an expression of vector algebra that already accounts for $\rho_s^b = \vec{n} \cdot \vec{P}$.

This now allows us to simplify greatly how we deal with the dipoles in materials:

$$\nabla \cdot \varepsilon_o \vec{E} = \text{all charges} = \rho_v + \rho_v^b$$

$$\uparrow \qquad \uparrow$$

placed charges   bound charges

$$\nabla \cdot \varepsilon_o \vec{E} = \rho_v - \nabla \cdot \vec{P} = \rho_v - \nabla \cdot \varepsilon_o \chi_e \vec{E}$$

Thus, $\nabla \cdot \left[\varepsilon_o \vec{E} + \vec{P} \right] = \rho_v$

$$\nabla \cdot \left[\varepsilon_o (1 + \chi_e) \vec{E} \right] = \rho_v \quad \Rightarrow \quad \nabla \cdot \vec{D} = \rho_v, \ D = \varepsilon_o \varepsilon_r E$$

$$\nabla \cdot \left[\varepsilon_o \varepsilon_r \vec{E} \right] = \rho_v$$