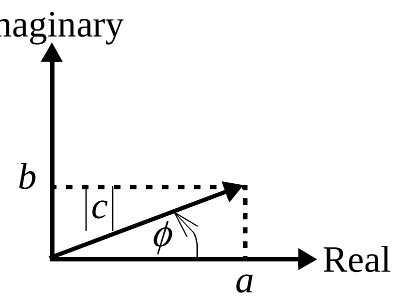
Review of Phasor Notation

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Complex Vectors and Time Harmonic Representation

(Chapter 1.5, 1.6 Text)

Complex Numbers



i) Rectangular/Cartesian Representation

$$c = a + jb$$

ii) Polar Representation

$$c = |c|e^{j\phi}, \text{ where } \quad |c| = \sqrt{a^2 + b^2}$$

$$\phi = \tan^{-1}\left(b/a\right)$$

$$c = a + jb = |c|e^{j\phi} = |c|\cos\phi + j|c|\sin\phi$$

real part   imaginary part   magnitude   phase

iii) Addition/Subtraction

Cartesian representation most convenient

$$c = a + jb$$
$$d = e + jh$$

Then: $c + d = (a+e) + j(b+h)$

$$c - d = (a-e) + j(b-h)$$

iv) Multiplication/Division

Note that $(j)^2 = -1$

$$cc^* = a^2 + b^2 = |c|^2,$$

where $c^* = a - jb$, the complex conjugate.

v) Cartesian Representation

$$cd = (a + jb)(e + jh) = (ae - bh) + j(ah + be)$$

$$\frac{c}{d} = \frac{a + jb}{e + jh} = \frac{a + jb}{e + jh}\,\frac{e - jh}{e - jh} = \frac{(ae + bh) + j(be - ah)}{e^2 + h^2}$$

vi) Polar Representation (more convenient for multiplication and division)

$$c = |c|e^{j\phi_1} \qquad d = |d|e^{j\phi_2} \qquad cd = |c||d|e^{j(\phi_1+\phi_2)} \qquad \frac{c}{d} = \frac{|c|}{|d|}e^{j(\phi_1-\phi_2)}$$

vii) Complex Representation of Time-harmonic Scalars

Consider:

$$v(t) = \underbrace{A}_{\text{amplitude}}\cos(\underbrace{\omega_o}_{\text{angular/radian frequency}} t + \underbrace{\phi}_{\text{phase}})$$

magnitude $|\hat{a}| = [\hat{a}\hat{a}*]^{1/2}$

and $\underbrace{\omega}_{\text{radian frequency}} = 2\pi \underbrace{f}_{\text{frequency}}$

amplitude $\mathrm{Re}[\hat{a}_1]$

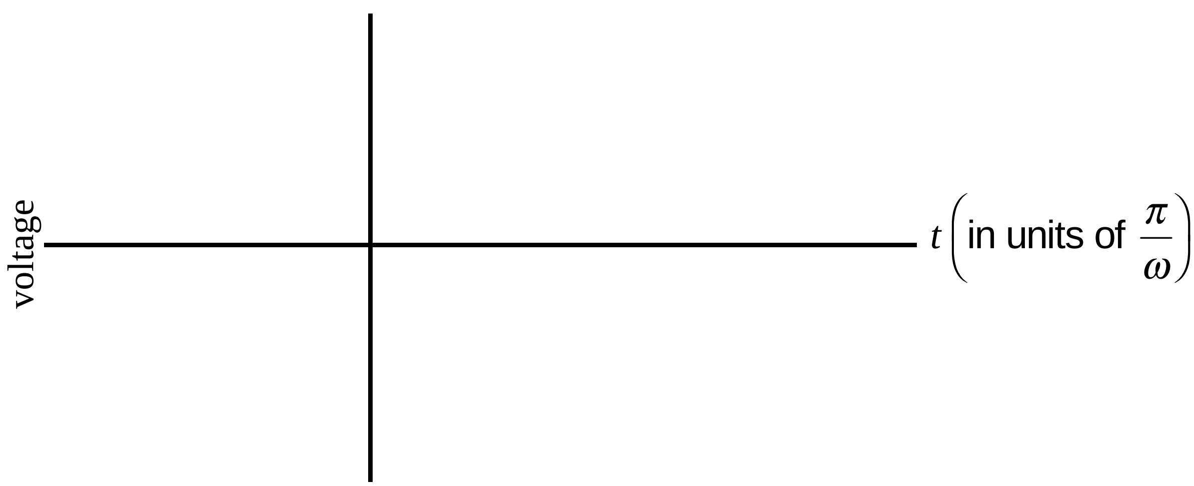
Sinusoidally Time Varying Fields (1.6 Text)

1) $f = A\cos(\omega t + \phi) = 110\cos\left(120\pi t + \frac{\pi}{4}\right)$

$$\left(\text{Note: } \frac{\pi}{4} \text{ radians } = 45°\right)$$

is a scalar field that depends on time only, but not on position (location). This is not a wave.

For example, this is the voltage in any electric household plug. Note that the phase ϕ simply alters the starting point.



The circular frequency $\omega = 2\pi f$, where f is the frequency in Hz. Here, $f = 60$ hertz (Hz) and $\omega = 120\pi = 377$ rad / sec

2) Waves -- travelling waves (main subject of this course)

$$v = V_o \cos\omega\left(t - \frac{z}{\mathrm{v}}\right)$$
$$= V_o \cos(\omega t - kz) \text{ or } V_o \cos(\omega t - \beta z)$$
$$= V_o \cos 2\pi\left(ft - \frac{z}{\lambda}\right)$$

$\boxed{\text{Repeats} \quad z \rightarrow z + \lambda}$

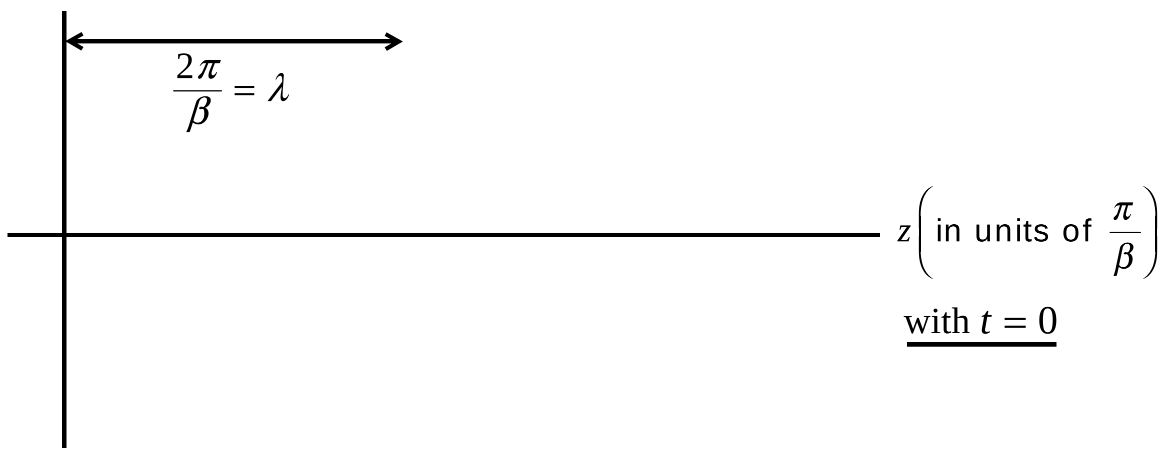
k (or β) $\equiv \dfrac{\omega}{\mathrm{v}} = \dfrac{2\pi f}{\mathrm{v}} = \dfrac{2\pi}{\lambda}$ propagation constant

$\mathrm{v} = \lambda f$

λ = wavelength

v = phase velocity

$T \equiv \dfrac{2\pi}{\omega} \equiv \dfrac{1}{f} \equiv$ period



This is a **scalar** traveling wave.

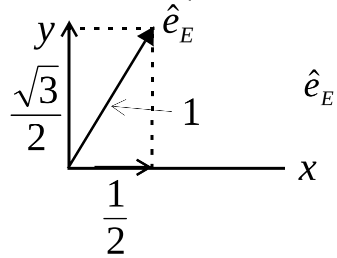
3) Vector Travelling Waves -- electromagnetic waves

Examples:

$$\text{(i)}\vec{E} = \hat{e}_x E_o \cos\omega\left(t - \frac{y}{\text{v}}\right) = \hat{e}_x E_o \cos(\omega t - ky)$$

(x-oriented field, $+y$ propagation)

$$\text{(ii)}\vec{E} = \left(\frac{1}{2}\hat{e}_x + \frac{\sqrt{3}}{2}\hat{e}_y\right) E_o \cos(\omega t - kz)$$



$$\hat{e}_E = \frac{1}{2}\hat{e}_x + \frac{\sqrt{3}}{2}\hat{e}_y$$

$$\text{(iii)}\vec{E} = \hat{e}_x E_o \cos\left(\omega t - kz + 45^\circ\right)$$

(x-oriented, $+z$ propagation)

$$\text{(iv)}\vec{H} = -\hat{e}_x H_o \sin\left(\frac{\pi}{a}x\right)\cos(\omega t - \beta z) + \hat{e}_z H_1 \cos\left(\frac{\pi}{a}x\right)\sin(\omega t - \beta z)$$

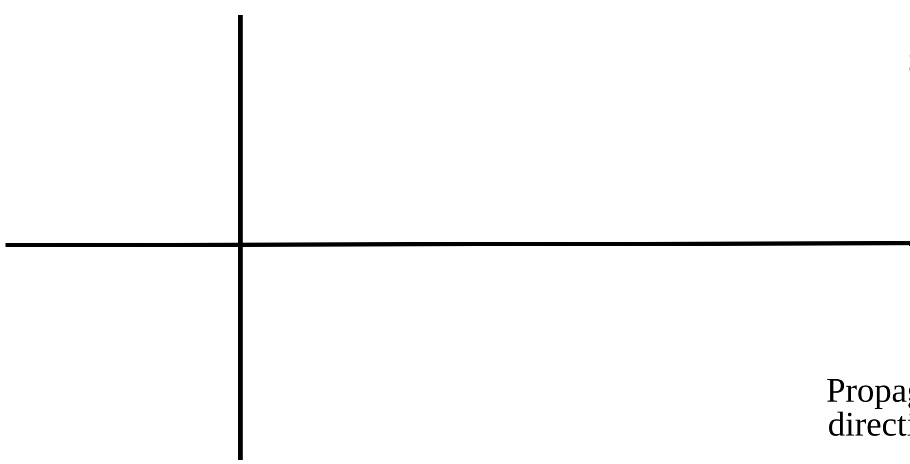
(waveguide example)

Example: (x-directed planewave, $-y$ propagation)

$$\vec{E}(y,t) = \hat{e}_x 100 \cos\left(\omega t + \beta y + 45^\circ\right)$$
$$= \hat{e}_x 100 \cos\left[\omega\left(t + \frac{y}{\text{v}}\right) + 45^\circ\right]$$

$$\frac{\omega}{\text{v}} = \beta$$

This is a vector travelling wave field.



$t = 0$

$t = \frac{\pi}{4\omega}$

$y\left(\text{in units of } \frac{\pi}{\beta}\right)$

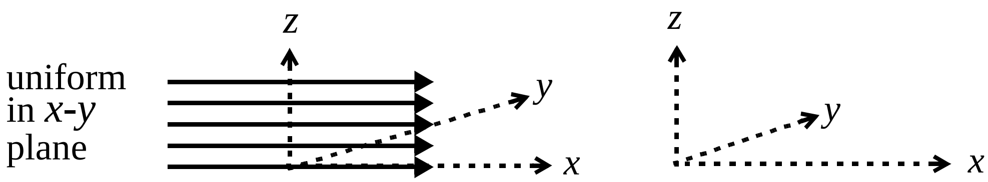
Propagates in the $-y$ direction

At $t = 0$: $\cos\left(\beta y + 45^\circ\right)$

At $t = \dfrac{\pi}{4\omega}$: $\cos\left(\beta y + 90^\circ\right) = -\sin\beta y$

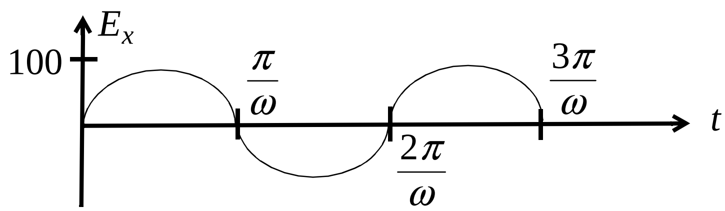
Note that this vector field:

(i) always points in the $\hat{e}_x$ direction (reverses, of course).
(ii) propagates in the y direction.
(iii) spatially depends only on the y coordinate (which is the direction of propagation).



The magnitude E, that is, the density of these lines (or the length of the arrows), varies sinusoidally, both in time and in the variable y direction of propagation.

At $y = -\dfrac{3}{8}\lambda$



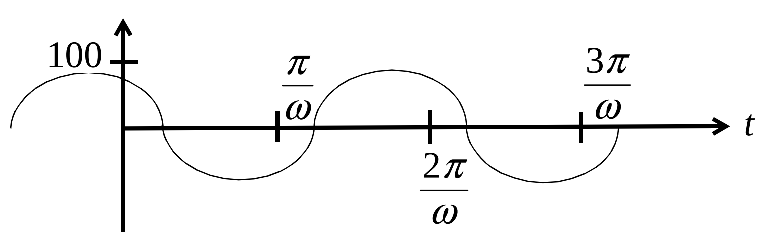
At $y = -\dfrac{3}{8}\lambda$:

$$\vec{E} = \hat{e}_x 100\cos\left(\omega t - \frac{2\pi}{\lambda}\frac{3}{8}\lambda + \frac{\pi}{4}\right)$$

$$= \hat{e}_x 100\cos\left(\omega t - \frac{\pi}{2}\right) = \hat{e}_x 100\sin\omega t$$

At $y = 0$:

$$\vec{E} = \hat{e}_x 100\cos\left(\omega t + \frac{\pi}{4}\right)$$



Method of Phasors ("Set your phasors on 'Stun,' Mr. Spock!")

Background material needed:

Review of complex algebra

Eulers Law: $e^{jx} = \cos x + j\sin x$ $\quad\quad$ $\mathbf{Re}\left[e^{jx}\right] = \cos x$, $\mathbf{Im}\left[e^{jx}\right] = \sin x$

Suppose we have a field that varies sinusoidally in time and/or space. For a scalar field, voltage, for example, we have:

$v = V_o \cos\omega t$ (a sinusoidal voltage)

$v = V_o \cos(\omega t - kz)$ (a voltage wave on a transmission line)

These are **real measurable** quantities.

We represent them by writing as follows:

$v = \mathbf{Re}\left\{V_o e^{j\omega t}\right\}$ Now we do either of (a) or (b):

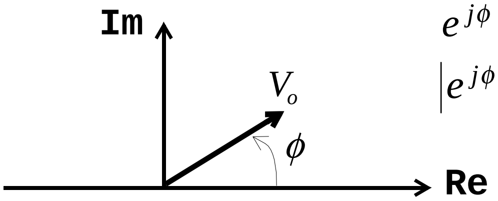
(a) Drop "$\mathbf{Re}$". The remaining $\mathbf{v} = V_o e^{j\omega t}$ is called the complex voltage (including time dependence) or voltage phasor (including time dependence).

(b) $v = \mathbf{Re}\left\{V_o e^{j\omega t}\right\}$ Drop "$\mathbf{Re}$" and "$e^{j\omega t}$"

The resultant $\mathbf{v} = V_o$ is called the (complex) phasor. Note that, for this example, the complex phasor happens to be real. (PHASORS are written in bold typeface.)

$$\text{If}\quad \mathbf{v} = V_o \cos(\omega t + \phi)$$
$$= \mathbf{Re}\left[V_o e^{j(\omega t + \phi)}\right] = \mathbf{Re}\left[V_o e^{j\omega t} e^{j\phi}\right]$$
$$= \mathbf{Re}\left[V_o e^{j\phi} e^{j\omega t}\right]$$

$V_o e^{j\phi}$ is a complex quantity in polar representation.



$$e^{j\phi} = \cos\phi + j\sin\phi$$
$$\left|e^{j\phi}\right| = \sqrt{\cos^2\phi + \sin^2\phi} = 1$$

Graphical Picture of Complex Fields

Consider a vector field with harmonic time dependence:

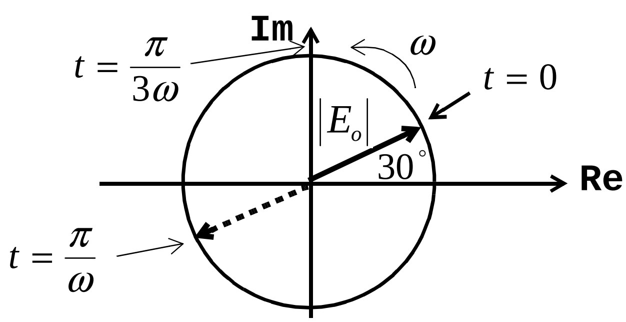
$$\vec{E} = \hat{e}_x E_o \cos(\omega t + 30^\circ) \equiv \mathbf{Re}\left[\hat{e}_x E_o e^{j\omega t} e^{j\frac{\pi}{6}}\right]$$

This is another notation for $\hat{e}_x E_o \angle 30^\circ e^{j\frac{\pi}{6}}$.

Complex form: $\vec{E} = \hat{e}_x E_o \angle 30^\circ e^{j\omega t}$

phasor amplitude (also a vector)

If we picture the entire time dependent complex field in the complex plane, we see that it rotates at angular velocity ω.



Note that the direction of the complex field on this complex plane has **no relationship** to its direction in real physical space, which here is constant $\hat{e}_x$.

The entire complex field, including $e^{j\omega t}$ time dependence, is often called "Rotating Phasor." The real field is the **projection** on the **Re** axis.

Amplitude, Time Average, Mean Square and Root Mean Square

(a) **Amplitude** is the maximum value of the time oscillation (real number).

(i) $\vec{E} = \hat{e}_x 100 \cos(\omega t - \beta z + \phi)$ real

$\mathbf{E} = \hat{e}_x 100 e^{j\phi}$ phasor

Amplitude is 100

Amplitude $\equiv$ magnitude of phasor $\equiv |\mathbf{E}| = \left[\vec{E}\,\vec{E}^*\right]^{1/2}$

(ii) $\mathbf{v} = 100 e^{-j\beta z} - 100\, e^{+j\beta z}$ 2 waves in opposite direction, phasor form

$$\left[\text{Use } \sin x = \frac{e^{jx} - e^{-jx}}{2j} \right]$$

$$= -\frac{200\, j}{2\, j}\left[e^{+j\beta z} - e^{-j\beta z}\right] = -\,200\, j \sin \beta z$$ $\Leftarrow$ This is the phasor (simplified).

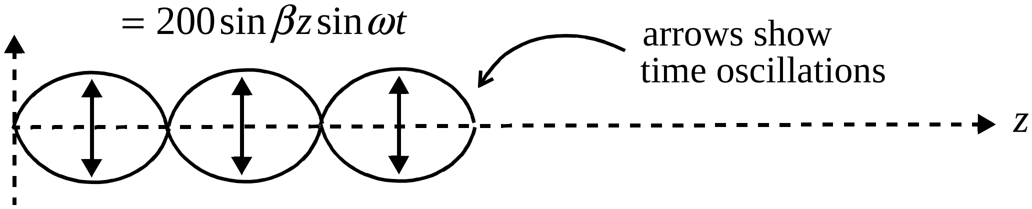
Amplitude of the **total** wave is

$$|\,\mathbf{v}\,| = |-200\, j \sin \beta z\,| = \sqrt{(-200\, j \sin \beta\, z)(200\, j \sin \beta\, z)} = 200 \sin \beta\, z$$

function of z!

Now, real form $v(z,t) = 200 \sin \beta z \sin \omega t$ because:

$$v\,(z,t) = \mathbf{Re}\left[-200\, j \sin \beta\, z\, e^{\,j\omega\, t}\right]$$
$$= \mathbf{Re}\left[-200\, j \sin \beta z \cos \omega t + 200 \sin \beta z \sin \omega t\right]$$
$$= 200 \sin \beta z \sin \omega t$$



(b) **Time average** of a harmonic quantity is always zero.

$$v(t) = V_o \cos(\omega t + \phi) \qquad \mathbf{v} = V_o e^{j\phi} \leftarrow \text{phasor}$$

$$\langle v(t) \rangle = \frac{1}{T}\int_0^T V_o \cos(\omega t + \phi) \equiv 0 \qquad T = \frac{1}{f}$$

(c) **Mean square** is not zero.

$$\langle v^2(t) \rangle = \frac{1}{T}\int_0^T V_o^2 \cos^2(\omega t + \phi)\, dt$$
$$= \frac{1}{T}\int_0^T V_o^2 \left[\frac{1}{2} + \frac{1}{2}\cos(2\,\omega t + 2\,\phi)\right] dt$$
$$= \frac{V_o^2}{2}$$

Using phasors for v^2 cannot give correct answers since phasors are **not valid for a nonlinear situation**. Try it and see:

$$v^2 = V_o^2 \cos^2 \omega t, \text{ but } \mathbf{v}^2 = V_o^2 e^{2j\omega t + 2j\phi}$$

Does not give $\cos^2$ when real part is taken for the time average. But the following "trick" works:

For the time average: (**Note**: It gives **only** the time average.)

$$\left\langle v^2(t) \right\rangle = \mathbf{Re}\,\frac{1}{2}\,\mathbf{v}\mathbf{v}^* = \mathbf{Re}\,\frac{1}{2}\,V_o e^{j\phi}\,V_o e^{-j\phi} = \frac{V_o^2}{2}$$

(This works for vector phasors, too!)

If: $\vec{\mathbf{E}} = \hat{e}_x E_x + \hat{e}_y E_y$

$$\left\langle \left| \vec{E}(t,z) \right|^2 \right\rangle = \mathbf{Re}\,\frac{1}{2}\left(\underset{\text{phasor}}{\underset{\uparrow}{\vec{\mathbf{E}}}} \cdot \underset{\uparrow}{\vec{\mathbf{E}}}^* \right)$$

Vector "dot" product

$$\vec{E} = E_o\, e^{-jkz}\, \hat{e}_x$$

The *Mean Square* is related to Average Power (averaged over whole cycles).

The same "trick" works for any other second order quantity.

$$\left\langle v(t)\, i(t) \right\rangle = \mathbf{Re}\,\frac{1}{2}\left[\mathbf{v}\,\mathbf{i}^* \right]$$

Note: The "**Re**" is necessary here, since v and i may have a phase difference.

(d) **Root Mean Square (RMS)**

$$\sqrt{\left\langle v^2(t) \right\rangle} \quad \text{or} \quad \sqrt{\left\langle \left| E(t) \right|^2 \right\rangle}$$

For harmonic fields, RMS value $= \dfrac{1}{\sqrt{2}}$ amplitude.

Example:

If: $\vec{E} = \hat{e}_r 100 \cos(\omega t - \beta z)$

$\vec{H} = \hat{e}_\phi 0.3 \cos(\omega t - \beta z - 60°)$

cylindrical coordinates (r, ϕ, z)

Calculate $\left\langle \vec{E} \times \vec{H} \right\rangle$, using **both** real fields and phasors.

Solution:

(a) Using real fields

$$\vec{E} \times \vec{H} = \hat{e}_z\, 30 \cos(\omega t - \beta z) \cos(\omega t - \beta z - 60°)$$

Use product of $\cos A \cdot \cos B = \frac{1}{2}\left\{\cos(A-B)+\cos(A+B)\right\}$

$$\vec{E}\times\vec{H} = \hat{e}_z 15\left[\cos 60° + \cos(2\omega t - 2\beta z - 60°)\right]$$

$$\left\langle \vec{E}\times\vec{H}\right\rangle = \hat{e}_z[7.5+0]$$

(b) Using phasors

$$\mathbf{\vec{E}} = \hat{e}_r 100$$

$$\left\langle \vec{E}\times\vec{H}\right\rangle = \frac{1}{2}\,\mathbf{Re}\left(\mathbf{\vec{E}}\times\mathbf{\vec{H}}^*\right) \qquad \mathbf{\vec{H}} = \hat{e}_\phi 0.3\, e^{-j\frac{\pi}{3}}$$

Phasors corresponding to the real $\vec{E}(t,z), \vec{H}(t,z)$

$$\frac{1}{2}\mathbf{E}\times\mathbf{H}^* = \hat{e}_z 15 \cdot e^{+j\frac{\pi}{3}}$$

$$\frac{1}{2}\mathbf{Re}\ \mathbf{E}\times\mathbf{H}^* = \hat{e}_z 15\cos 60° = \hat{e}_z 7.5$$

Phasor examples with waves

(1) Space dependence is due to propagation only

(a) Uniform plane EM wave -- vector

(b) Transmission line wave -- scalar voltage or current

(1a) Propagation is only in **one** direction

Real field: $\vec{E}(z,t) = \hat{e}_x E_o \cos\left\{\omega\left(t-\frac{z}{v}\right)+\phi\right\} = \hat{e}_x E_o \cos(\omega t - \beta z + \phi)$

Propagation is in $+\hat{z}$ direction $\quad \beta = \frac{\omega}{\mathrm{v}} \quad \mathrm{v} = \frac{\omega}{\beta}$

Complex field: $\vec{E}(z,t) = \hat{e}_x E_o e^{j(\omega t - \beta z + \phi)}$

$$= \hat{e}_x E_o e^{j\phi} e^{j(\omega t - \beta z)}$$

Phasor field: (two options possible)

(i) Drop entire $e^{j(\omega t - \beta z)}$ dependence

Phasor: $\mathbf{\vec{E}} = \hat{e}_x E_o e^{j\phi}$ (algebraic) $\equiv \hat{e}_x E_o \angle\phi$ (schematic)

For example, $\mathbf{\vec{E}} = \hat{e}_x 100\angle 30^\circ \equiv \hat{e}_x 100\, e^{j30°}$

$$\vec{E}(t,z) = \mathbf{Re}\left[\hat{e}_x 100\, e^{j\left(\omega t - \beta z + \pi/6\right)}\right]$$

$$= \hat{e}_x 100\cos(\omega t - \beta z + 30°)$$

Another example:

$$\left(j = \cos\phi + j\sin\phi, \text{ when } \phi = 90°\right)$$

$$\vec{\mathbf{E}} = \hat{e}_y\, j\, 50 = \hat{e}_y\, 50 \angle 90°$$

$$\vec{E}(t,z) = \mathbf{Re}\left[\hat{e}_y\, 50\, j\, e^{j(\omega t - \beta z)}\right] = \hat{e}_y\, \mathbf{Re}\left[50\, e^{j(\omega t - \beta z + \pi/2)}\right]$$

$$= -50\,\hat{e}_y \sin(\omega t - \beta z) = 50\,\hat{e}_y \cos(\omega t - \beta z + 90°)$$

Another example: $\vec{\mathbf{E}} = \left(\hat{e}_x - j\,\hat{e}_y\right)100$

$$\vec{E}(t,z) = \mathbf{Re}\left[100\left(\hat{e}_x - j\,\hat{e}_y\right)e^{j(\omega t - \beta z)}\right]$$

$$= 100\left[\hat{e}_x \cos(\omega t - \beta z) + \hat{e}_y \sin(\omega t - \beta z)\right]$$

$$\left\langle \left|\mathbf{e}\right|^2 \right\rangle = \mathbf{Re}\left[\tfrac{1}{2}\vec{E}\cdot\vec{E}^*\right] = \tfrac{1}{2}(100)^2\left(\hat{e}_x - j\,\hat{e}_y\right)\cdot\left(\hat{e}_x + j\,\hat{e}_y\right)$$

$$= \tfrac{1}{2}(100)^2(1+1) = (100)^2$$

(ii) Drop only $e^{j\omega t}$ dependence

$$\vec{E}(z,t) = \hat{e}_x\, E_o \cos(\omega t - \beta z + \phi)$$

$$\vec{E}(z,t) = \hat{e}_x\, \mathbf{Re}\left[E_o\, e^{j(\omega t - \beta z + \phi)}\right]$$

$$\vec{\mathbf{E}} = \hat{e}_x\, E_o\, e^{-j\beta z + j\phi}$$

We can see that $-\beta z$ and ϕ both are phases. The phase changes with propagation. The phase **rotates** in the complex plane.

For waves travelling in **one** direction only, we don't usually use this. Instead, we drop off of $e^{j(\omega t - \beta z)}$ -- usually (but not always).

Example:

(1b) Transmission line wave

Waves travelling in **two** opposite directions

$$v(t,z) = V_o \cos(\omega t - \beta z) + \Gamma\, V_o \cos(\omega t + \beta z + \phi)$$

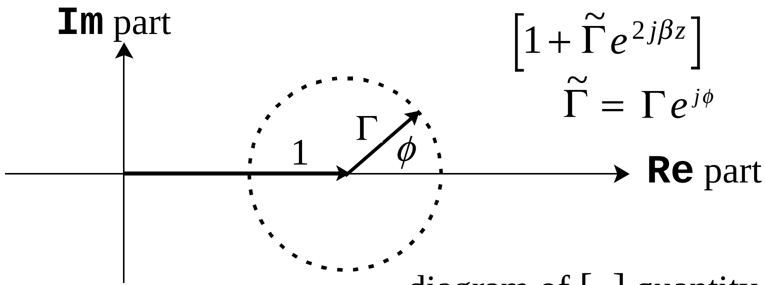
$$v(t,z) = \mathbf{Re}\left[V_o\, e^{-j\beta z} e^{j\omega t} + \Gamma\, V_o\, e^{j\phi} e^{+j\beta z} e^{j\omega t}\right]$$

We only have one option: Drop $e^{j\omega t}$ dependence, since βz dependence is not the same on the two parts.

Phasor: $\mathbf{v} = V_o\left[e^{-j\beta z} + \Gamma\, e^{j\phi} e^{+j\beta z}\right]$

$$= V_o\, e^{-j\beta z}\left[1 + \tilde{\Gamma}\, e^{2j\beta z}\right]$$

$\tilde{\Gamma} = \Gamma\, e^{j\phi}$ = complex reflection coefficient



$$\left[1+\tilde{\Gamma}e^{2j\beta z}\right]$$

$$\tilde{\Gamma} = \Gamma e^{j\phi}$$

called “Crank Diagram”

diagram of [ ] quantity in complex plane

(2) **Space dependence may exist perpendicular to the direction of propagation**; for example, laser beams, guided electromagnetic waves, optical waves in fibers. Not the subject of ECE 130A, but used in ECE 130B.

Example of (2) with (i) propagation in one direction: E.M. wave in rectangular waveguide

$$\mathbf{E}(x,y,z,t) = \hat{e}_y\, E_o \sin\left(\frac{\pi}{a}x\right) e^{j(\omega t-\beta z)} \qquad x<a$$

Drop $e^{j(\omega t-\beta z)}$:

$$\mathbf{E} = \hat{e}_y\, E_o \sin\left(\frac{\pi}{a}x\right) \qquad x<a$$

Another example:

$$\mathbf{H} = -\hat{e}_x\, H_o \sin\left(\frac{\pi}{a}x\right) - j\,\hat{e}_z\, H_1 \cos\left(\frac{\pi}{a}x\right) \qquad x<a$$

Phasor:

Significance of j: $\hat{x}$ and $\hat{z}$ components are 90° out of phase

Real field is:

$$\vec{H} = -\hat{e}_x\, H_o \sin\left(\frac{\pi}{a}x\right)\cos(\omega t-\beta z) + \hat{e}_z\, H_1 \cos\left(\frac{\pi}{a}x\right)\sin(\omega t-\beta z)$$

Similar fields exist in any guided wave, such as waves in **optical fibers**!

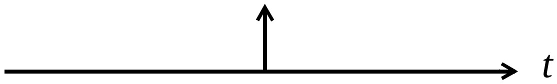
(3) **Modulated (time dependent amplitude) waves**

All the examples brought in (1) and (2) were waves of a single sinusoidal frequency and amplitude constant in time.

In communications, we impress **information** onto a single frequency wave by modulating its amplitude in **time**. The original sinusoid is the **carrier** and the modulation of the amplitude contains the information.

Example: Pulse code modulation

unmodulated carrier



$$\vec{E} = \hat{e}_x\, E_o \cos\omega\left(t - \frac{z}{v}\right)$$

$z = 0$

a single pulse impressed onto a carrier

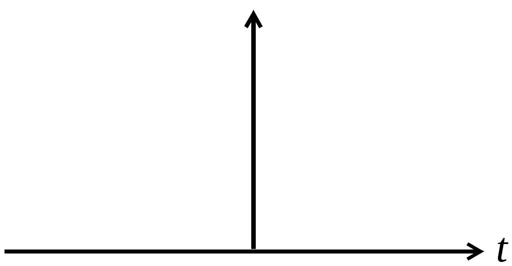
$$\vec{E} = \hat{e}_x\, E_o\, e^{-\left(t - z/v\right)^2 / 2\sigma_o^2} \cos\left(\omega_o t - \frac{z}{v}\right)$$

At z = 0: $\vec{E} = \hat{e}_x\, E_o\, e^{-t^2/2\sigma_o^2} \cos\omega_o t$

Phasor form: $\vec{E} = \hat{e}_x\, E_o\, e^{-t^2/2\sigma_o^2}\, e^{j\omega_o t}$

amplitude envelope carrier

This envelope function is a Gaussian.

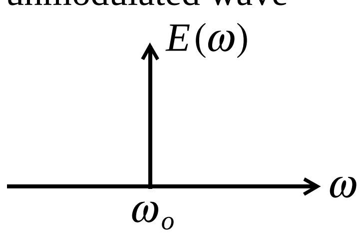


$$\left|E\right|^2 = E \cdot E^* = E_o^2\, e^{-t^2/\sigma_o^2}$$

The modulated wave is no longer a single frequency. By Fourier analysis, it is easy to show that a **band** of frequencies about ω_o are present.

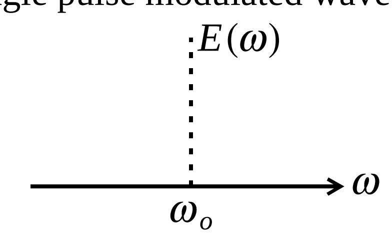
Frequency Domain Representation

unmodulated wave



$$E(t, z=0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} E(\omega)\, e^{j\omega t}\, d\omega$$

a single pulse modulated wave



$$E(\omega) = \int_{-\infty}^{+\infty} E(t)\, e^{-j\omega t}\, dt$$

For the single pulse modulated field: $E(\omega) = \sqrt{2\pi}\,\sigma_o E_o e^{-\frac{\sigma_o^2}{2}(\omega-\omega_o)^2}$

$$\left|E(\omega)\right|^2 = 2\pi\,\sigma_o^2 E_o^2 e^{-\sigma_o^2(\omega-\omega_o)^2}$$

Such a pulse is in fact a very good approximation to an actual one travelling in an optical fiber part of an optical communication system.

Schematic of such a system: