Electrostatic Problems with Dielectrics

$$C = \frac{Q}{V} = \frac{A}{d_2/\varepsilon_2 + d_1/\varepsilon_1}$$

Examples of Electrostatic Problems with Dielectrics

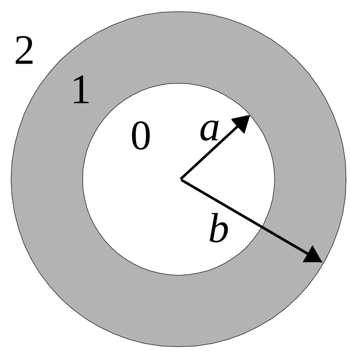
Problem: Find $\vec{D}$ (electric flux density), $\vec{E}$ (electric field intensity), and $\vec{P}$ (polarization) for a metallic sphere (radius a, charge Q), coated by a dielectric (radius b), and the charge densities at the interfaces.

Solution: Use Gauss' Law $\oint_s \vec{D} \cdot d\vec{s} = \int_v \rho_v \, dv$

In *region* 0, $r < a$, $\vec{D} = 0$, $\vec{E} = 0$

In *region* 1, $a < r < b$:

$$\vec{D}_1 = \hat{e}_r \frac{Q}{4\pi r^2}$$



In *region* 2, $r > b$:

$$\vec{D}_2 = \hat{e}_r \frac{Q}{4\pi r^2}$$

The electric fields are:

In *region* 1, $a < r < b$:

$$\vec{E}_1 = \hat{e}_r \frac{Q}{4\pi \varepsilon_1 r^2}$$

In *region* 2, $r > b$:

$$\vec{E}_2 = \hat{e}_r \frac{Q}{4\pi \varepsilon_o r^2}$$

Note that the electric field varies with the medium, and the flux density does not.

The polarization is:

$$\vec{P} = \varepsilon_o \chi_e \vec{E} \text{ and } \varepsilon_r = 1 + \chi_e$$

In *region* 1:

$$\vec{P}_1 = \varepsilon_o \chi_e \vec{E}_1 = \varepsilon_o (\varepsilon_r - 1) \vec{E}_1 = \hat{e}_r \frac{\varepsilon_o (\varepsilon_r - 1) Q}{4\pi \varepsilon_1 r^2}$$

In *region* 2: $\vec{P}_2 = 0$

Now the surface charge on $0-1$ boundary. The boundary condition is:

$$\vec{n}\cdot\left(\vec{P}_1-\vec{P}_o\right) = -\rho_{ps} ,$$

$$\text{or} \quad \hat{e}_r\cdot\left[\hat{e}_r\,\frac{\varepsilon_o\left(\varepsilon_r-1\right)Q}{4\pi\,\varepsilon_1\,a^2}-1\right] = -\rho_{ps}$$

Then:

$$\rho_{ps}\Big|_{r=a} = -\frac{\varepsilon_o\left(\varepsilon_r-1\right)Q}{4\pi\,\varepsilon_1\,a^2}$$

Now for $r=b$:

$$\hat{e}_r\cdot\left[0-\hat{e}_r\,\frac{\varepsilon_o\left(\varepsilon_r-1\right)Q}{4\pi\,\varepsilon_1\,b^2}\right] = -\rho_{ps}$$

Then:

$$\rho_{ps}\Big|_{r=b} = \frac{\varepsilon_o\left(\varepsilon_r-1\right)Q}{4\pi\,\varepsilon_1\,b^2}$$

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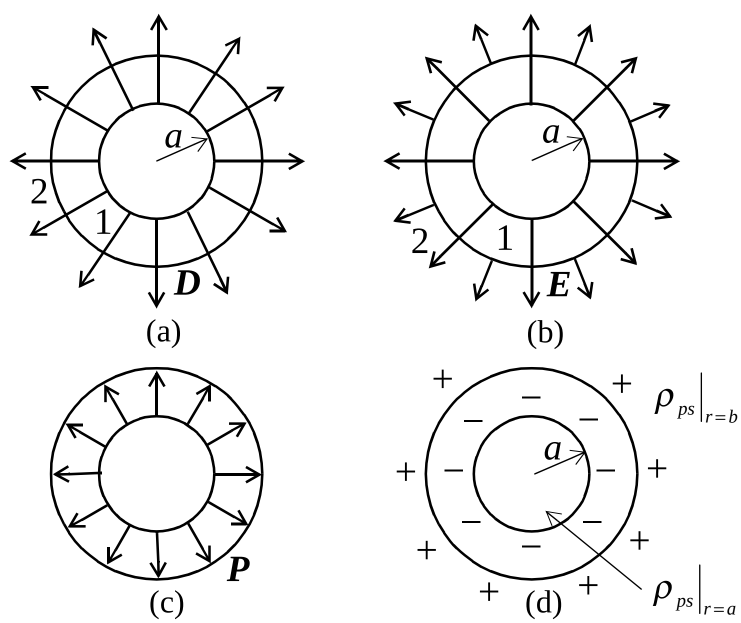
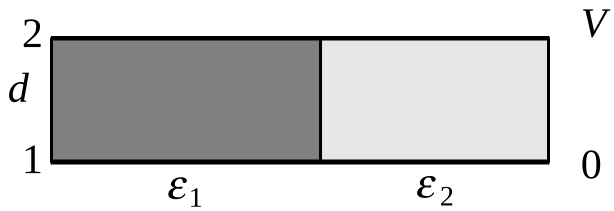


(a) The electric flux density in all regions where it may be seen that $\vec{D}$ is continuous. (b) The electric field intensity $\vec{E}$ in all regions where it may be seen that $\vec{E}$ is larger in region 2 and hence was presented by closer flux lines. (c) The polarization $\vec{P}$ that only exists in region 1 where we have dielectric material and (d) the polarization surface charge on both surfaces of region 1.

Example

Parallel plate capacitor with two different dielectrics:



Determine (1) the electric field intensities and the electric flux density vectors, and (2) the surface charge density of polarization charges.

(1) $\varepsilon_1 = \varepsilon_o \varepsilon_{r1}$

The electric field is: $V = \int_0^d \vec{E} \cdot d\vec{l} = E d$ or $E = V/d$, everywhere

$\vec{D}_1 = \varepsilon_1 \vec{E}_1$ (*region* 1) and $\vec{D}_2 = \varepsilon_2 \vec{E}_2$ (*region* 2)

Or, in terms of V:

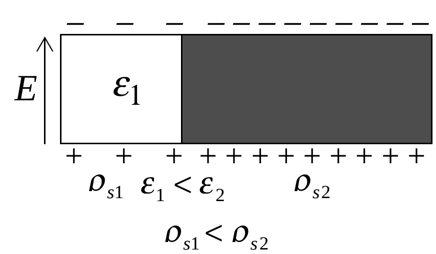
$$\vec{D}_1 = \varepsilon_1 \frac{V}{d} \quad \text{and} \quad \vec{D}_2 = \varepsilon_2 \frac{V}{d}$$

(2) At the surface of the conductor, the surface free charge density is:

$$\rho_s = D_n$$

Therefore, in *region* 1: $\rho_{s1} = D_{n1} = \varepsilon_1 \frac{V}{d}$

and in *region* 2: $\rho_{s2} = D_{n2} = \varepsilon_2 \frac{V}{d}$



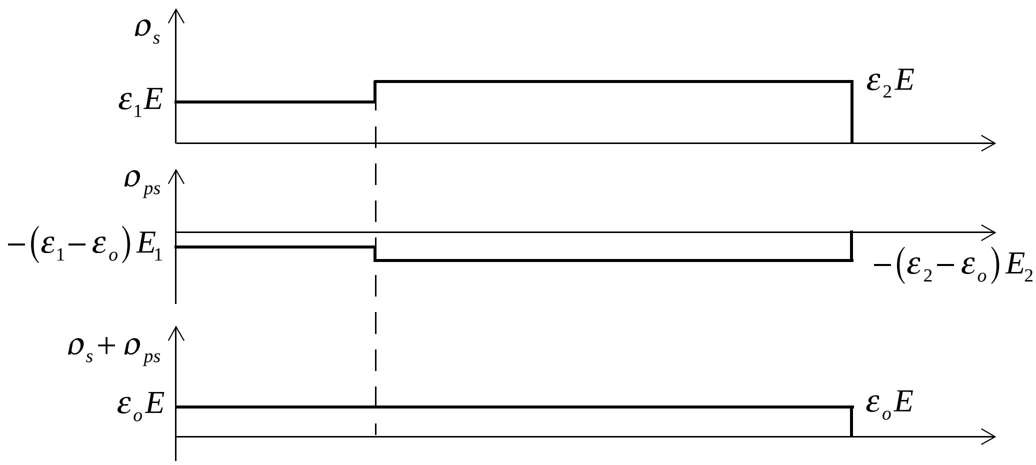
The surface charge density of polarization charge is:

$-\rho_{ps1} = p_{normal1} \rightarrow$ since $\vec{P} = \varepsilon_o \chi_e \vec{E}$ and $\varepsilon_{r1} = 1 + \chi_{e1}$

$\Rightarrow \vec{P}_1 = \varepsilon_o (\varepsilon_{r1} - 1) \vec{E}_1$, or $\vec{P}_1 = (\varepsilon_1 - \varepsilon_o) \frac{V}{d} = -\rho_{ps1}$

and for *region* 2:

$\Rightarrow \vec{P}_2 = \varepsilon_o (\varepsilon_{r2} - 1) \vec{E}_1$, or $\vec{P}_2 = (\varepsilon_2 - \varepsilon_o) \frac{V}{d} = -\rho_{ps2}$



Free surface charge density distribution ρ_s and the polarization charge density distribution ρ_{ps} on the interface between the conducting planes and the dielectric materials. The total charge density on the surface of the conductor is constant and equal to $\varepsilon_o E$.

The total charge density (sum of bound and free) is a constant $= \varepsilon_o E$.

Capacitance (We need this for transmission lines!)

A measure of the amount of charge a particular arrangement of two conductors is able to retain per unit voltage: $C \equiv Q/V$.

Example: Spherical capacitor

$$\oint_s \vec{D} \cdot d\vec{s} = \int_v \rho_v \, dv$$

Since $\vec{D} = \varepsilon \vec{E}$ and the total charge on the inner sphere is Q, we have:

$$\vec{E} = \frac{Q \hat{e}_r}{4\pi \varepsilon r^2}$$

and the potential difference is: $V_{ab} = -\int_b^a \vec{E} \cdot d\vec{l} = \int_a^b \vec{E} \cdot d\vec{l}$

Discussion: The minus sign indicates that positive work must be done to move a positive test charge from b to a against the field direction (radially out of a to b). Now, because of symmetry, $d\vec{l} = dr\,\hat{e}_r$.

Then,

$$V_{ab} = -\int_a^b \left[\frac{Q(\hat{e}_r \cdot \hat{e}_r)}{4\pi \varepsilon r^2} \right] dr = \left[-\frac{Q}{4\pi \varepsilon r} \right]_a^b$$

$$= \left(-\frac{Q}{4\pi \varepsilon} \right) \left(-\frac{1}{a} + \frac{1}{b} \right) = \frac{Q}{4\pi \varepsilon} \left(\frac{1}{a} - \frac{1}{b} \right)$$

The capacitance is:

$$C \equiv \frac{Q}{V} = \frac{4\pi \varepsilon}{1/b - 1/a}$$

Another example: Infinitely wide parallel plate capacitor.
Find capacitance/unit area.

There are two fields, E_1 and E_2, both directed normal to the plates.

Gauss' Law gives: $E_1 = \frac{\rho_s}{\varepsilon_1}$ and $E_2 = \frac{\rho_s}{\varepsilon_2}$.

The flux density $\vec{D}$ is (and must be) continuous across the dielectric boundary, and its value equals the surface charge density.

The potential is given by:

$$V = -\int_{d_2}^{0} E_2 \cdot d\vec{l} - \int_{d_1 + d_2}^{d_2} E_1 \cdot d\vec{l} = \frac{\rho_s d_2}{\varepsilon_2} + \frac{\rho_s d_1}{\varepsilon_1}$$

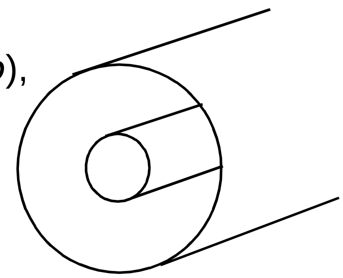
$$= \rho_s \left(\frac{d_2}{\varepsilon_2} + \frac{d_1}{\varepsilon_1} \right)$$

Now $Q = \rho_s A$.

So the capacitance is: $C = \frac{Q}{V} = \frac{A}{d_2/\varepsilon_2 + d_1/\varepsilon_1}$

Another important example:

a coaxial line (inner radius a, outer b), filled with dielectric



Again, Gauss' Law:

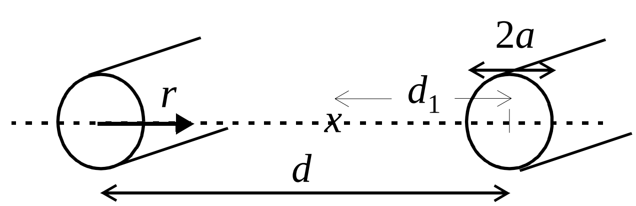
$$\oint_s \varepsilon_1 \vec{E} \cdot d\vec{s} = \int_v \rho_v \, dv$$

or for $a < r < b$: $\varepsilon_1 E 2\pi r L = \rho_s 2\pi a L$, or $E_r = \dfrac{\rho_s a}{\varepsilon_1 r}$

The voltage is $V = -\int_b^a E \cdot d\vec{l} = -\dfrac{\rho_s a}{\varepsilon_1} \int_b^a \dfrac{dr}{r} = \dfrac{\rho_s a}{\varepsilon_1} \ln\left(\dfrac{b}{a}\right)$,

so that $C = \dfrac{2\pi a \rho_s}{V} = \dfrac{2\pi\varepsilon_1 / \ln\left(b/a\right)}{unit\ length}$.

Another important example Two conductor transmission line.



There are equal and opposite charges per unit length on the conductors. Let $+r_l$ be on the right conductor. We will use the principal of superposition to solve this problem.

Gauss' Law for the right hand wire yields for the field:

$$\int_s \varepsilon \vec{E}_1 \cdot d\vec{s} = \int_c r_l \, dl$$

$$\int_{\phi=0}^{2\pi} \int_{z=0}^{l} \varepsilon \vec{E}_{1r} \hat{e}_r \cdot r \, d\phi \, dz \, \hat{e}_r = r_l \, l$$

$$E_{1r} = \frac{r_l}{2\pi \varepsilon r}$$

The electric field at d_1 due to right hand wire is

$$\vec{E}_1 = \frac{r_l}{2\pi \varepsilon d_1} \hat{e}_r$$

Now, the left hand wire:

$$\vec{E}_2 = \frac{r_l \left(-\hat{e}_r\right)}{2\pi \varepsilon \left(d - d_1\right)}$$

The total field is:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \left[-\frac{r_l}{2\pi \varepsilon \left(d - d_1\right)} + \frac{r_l}{2\pi \varepsilon d_1} \right]$$

$$V = -\int_{d-a}^{a} \left[\frac{r_l}{2\pi \varepsilon r} - \frac{r_l}{2\pi \varepsilon (d-r)} \right] \hat{e}_r \cdot \, d r \, \hat{e}_r = -\int_{d-a}^{a} \vec{E} \cdot d\vec{l}$$

C=?