Synthetic quantum field theory

Talk at

Can. Math. Soc. Summer Meeting 2013

Progress in Higher Categories

Urs Schreiber

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I) Brief Introduction and Overview

(continue reading)

II) Survey of some more details

(keep reading after the introduction)

Hilbert's 6th problem

David Hilbert, ICM, Paris 1900:

Mathematical Problem 6:

To treat [...] by means of **axioms**, *those* **physical sciences** *in which mathematics plays an important part*

[...] try first by a **small number of axioms** *to include as large a class as possible of physical phenomena, and then by adjoining new axioms to arrive gradually at the more special theories.*

[...] take account not only of those theories coming near to reality, but also, [...] of all **logically possible theories**.

Partial Solutions to Hilbert's 6th problem – I) traditional

	physics	maths
	prequantum physics	*differential geometry*
18xx-19xx	mechanics	symplectic geometry
1910s	gravity	Riemannian geometry
1950s	gauge theory	Chern-Weil theory
2000s	higher gauge theory	differential cohomology
	quantum physics	*noncommutative algebra*
1920s	quantum mechanics	operator algebra
1960s	local observables	co-sheaf theory
1990s-2000s	local field theory	(∞, n)-category theory

(table necessarily incomplete)

Partial Solutions to Hilbert's 6th problem – II) synthetic

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1. Foundation of **mathematics** in topos theory
("ETCS" [Lawvere 65]).

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by **"synthetic"** formulation:

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> $\left\{ \begin{array}{c} \text{Impose properties on} \\ \text{Add axioms to} \end{array} \right\}$ a $\left\{ \begin{array}{c} \underline{\text{topos}} \\ \underline{\text{intuitionistic type theory}} \end{array} \right\}$
>
> which ensure that the $\left\{ \begin{array}{c} \text{objects} \\ \text{types} \end{array} \right\}$ have
>
> structure of *differential geometric spaces*.
>
> ---
>
> Then formalize physics by $\left\{ \begin{array}{c} \underline{\text{universal constructions}} \\ \underline{\text{natural deduction}} \end{array} \right\}$.

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structure of *differential geometric spaces*.

$$\text{Then formalize physics by } \left\{ \begin{array}{c} \text{universal constructions} \\ \text{natural deduction} \end{array} \right\}.$$

- *Categorical dynamics* [Lawvere 67]
- *Toposes of laws of motion* [Lawvere 97]
- *Outline of synthetic differential geometry* [Lawvere 98]

But

modern fundamental physics

and

modern foundational maths

are both deeper

than what has been considered in these results...

$\longrightarrow$

Modern natural foundations.

Reconsider Hilbert's 6th in view of modern foundations.

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Modern foundations of fundamental physics is:

local Lagrangian boundary-/defect- quantum gauge field theory

(a recent survey is in [Sati-Schreiber 11])

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$$\boxed{\underline{\text{local}}\ \underline{\text{Lagrangian}}\ \underline{\text{boundary-/defect-}}\ \underline{\text{quantum}}\ \underline{\text{gauge field}}\ \underline{\text{theory}}}$$

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Modern foundations of mathematics is:

$$\boxed{\left\{\begin{array}{c}\underline{\text{homotopy type theory}},\\ \underline{\infty\text{-topos theory}}\end{array}\right\} + \underline{\text{internal } (\infty, n)\text{-category theory}}}$$

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Claim

In $\left\{\begin{array}{c}\textit{homotopy type theory}\\ \infty\textit{-topos theory}\end{array}\right\}$*-foundations*

fundamental physics is synthetically axiomatized

1. *naturally – the axioms are simple, elegant and meaningful;*
2. *faithfully – the axioms capture deep nontrivial phenomena* →

Project

This is an ongoing project involving joint work with

- Domenico Fiorenza
- Hisham Sati
- Michael Shulman
- Joost Nuiten

and others:

Differential cohomology in a cohesive ∞-topos [Schreiber 11].

You can find publications, further details and further exposition at:

http://ncatlab.org/schreiber/show/differential+cohomology+in+a+cohesive+topos

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Contents

(higher) gauge-

Lagrangian-

local

(bndry-/defect)-

quantum-

field theory

Contents

	physics	maths	
1)	(higher) gauge-	∞-topos theory, homotopy type theory	field theory
	Lagrangian-		
	local		
	(bndry-/defect)-		
	quantum-		

Contents

	physics	maths	
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2)	Lagrangian-	$\left\{\begin{array}{l}\text{differential cohomology,}\\ \text{cohesion modality}\end{array}\right.$	field theory
	local		
	(bndry-/defect)-		
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Remark. No approximation: non-perturbative QFT.

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Selected **examples and applications**:

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Ex1 Holographic quantization of Poisson manifolds and D-branes.

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Ex1 Holographic quantization of Poisson manifolds and D-branes.

Ex2 topol. ∞-YM $\overset{\text{bdr}}{\to}$ ∞-CS $\overset{\text{dfct}}{\to}$ ∞-WZW $\overset{\text{dfct}}{\to}$ ∞-Wilson surf.

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Ex3 Super p-branes, e.g. M5 ($\overset{\text{event.}}{\rightsquigarrow}$ Khovanov, Langlands, ...)

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Synthetic QFT Axioms

Synthetic QFT Axioms

(QFT 1) Gauge principle. *Spaces of physical fields are higher moduli stacks:*

$$\left\{ \begin{array}{c} \textit{objects} \\ \textit{types} \end{array} \right\} \textit{ of an } \left\{ \begin{array}{c} \infty\textit{-topos} \\ \textit{homotopy type theory} \end{array} \right\} \mathbf{H}.$$

$$\boxed{\mathbf{Fields} \in \mathbf{H}}$$

We discuss this in more detail below in <u>**1)**</u>.

Synthetic QFT Axioms

(QFT 2) Differential geometry.

The $\left\{\begin{array}{c}\infty\text{-topos} \\ \text{homotopy type theory}\end{array}\right\}$ carries two adjoint triples of $\left\{\begin{array}{c}\text{idempotent } \infty\text{-(co-)monads} \\ \text{higher modalities}\end{array}\right\}$ that equip $\left\{\begin{array}{c}\text{objects} \\ \text{types}\end{array}\right\}$ with "differential cohesive" geometric structure.

$$\boxed{\begin{array}{ccccc} \Pi & \dashv & \flat & \dashv & \sharp \\ \mathrm{Red} & \dashv & \Pi_{\mathrm{inf}} & \dashv & \flat_{\mathrm{inf}} \end{array}}$$

This is a joint refinement to homotopy theory of Lawvere's "synthetic differential geometry" and "axiomatic cohesion" [Lawvere 07] .

We discuss this in more detail in 2) below.

Synthetic QFT Axioms

Theorem

Differential cohesion in homotopy theory implies the existence of differential coefficient $\left\{ \begin{array}{c} \textit{objects} \\ \textit{types} \end{array} \right\}$ *modulating cocycles in differential cohomology.*

$$\begin{array}{ccccc} \flat \mathbf{B}\mathbb{G} & \xrightarrow{\text{include}} & \mathbf{B}\mathbb{G}_{\mathrm{conn}} & \xrightarrow{\text{forget}} & \mathbf{B}\mathbb{G} \\ \text{flat} & & \text{principal} & & \text{principal} \\ \infty\text{-connections} & & \infty\text{-connections} & & \infty\text{-bundles} \end{array}$$

Remark

This is absolutely not the case for differential cohesion interpreted non-homotopically.

Whence the title "*Differential cohomology in a cohesive ∞-topos*" [Schreiber 11].

Synthetic QFT Axioms

(QFT 3) Local Lagrangian action functionals.

The $\left\{ \begin{array}{c} \text{slice objects} \\ \text{dependent types} \end{array} \right\}$ over differential coefficients

$$\boxed{\begin{tikzcd} \mathbf{Fields} \arrow[d, "\exp(iS)"] \\ \mathbf{B}^n \mathbb{G}_{\mathrm{conn}} \end{tikzcd}}$$

are the *local action functionals*.

The $\left\{ \begin{array}{c} \text{correspondence spaces} \\ \text{relations} \end{array} \right\}$

$$\boxed{\begin{tikzcd} & \mathbf{Fields} \arrow[dl] \arrow[dr] & \\ \mathbf{Fields}_1 \arrow[dr, "\exp(iS_1)"'] & \arrow[l, phantom, "\Leftarrow"] & \mathbf{Fields}_2 \arrow[dl, "\exp(iS_2)"] \\ & \mathbf{B}\mathbb{G}_{\mathrm{conn}} & \end{tikzcd}}$$

are the *field trajectories*,
the *quantum observables*,
and the *defect- and boundary conditions*.

We discuss this in more detail below in **3)**.

Synthetic QFT Axioms

(QFT 4) Quantization.
Quantization is the passage to the "motivic" abelianization of these $\left\{ \begin{array}{c} \text{corespondence spaces} \\ \text{relations} \end{array} \right\}$ of $\left\{ \begin{array}{c} \text{slice objects} \\ \text{dependent types} \end{array} \right\}$ over the differential coefficients.

$$\begin{tikzcd} \mathrm{Bord}_n^{\mathrm{bdr}} \arrow[r, "\exp(iS)"'] \arrow[rr, bend left, "\int D\phi\, \exp(iS(\phi))"] & \mathrm{Corr}_n(\mathbf{H}, \mathbf{B}^n U(1)) \arrow[r, "\int D\phi(-)"'] & \mathrm{Mot}_n(\mathbf{H}) \end{tikzcd}$$

We discuss this in more detail below in **4)**.

This is established for 2-dimensional theories and their holographic 1-d boundary theories (quantum mechanics) by **Ex1** below. For higher dimensions this is a proposal for a systematic perspective.

End

of overview.

→ back to project page

→ on to further details

1)
Higher gauge field theory
∞-Topos theory
Homotopy type theory

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From the gauge principle to higher stacks.

Central principle of modern fundamental **physics** –
the gauge principle:

- Field configurations may be different and yet *gauge equivalent*.
- Gauge equivalences may be different and yet *higher gauge equivalent*.
- Collection of fields forms *BRST complex*, where (higher) gauge equivalences appear as (higher) ghost fields.

This means that moduli spaces of fields are

$$\textbf{geometric homotopy types} \quad \simeq \quad \textbf{higher moduli stacks}$$
$$\simeq \textbf{objects of an } \infty\textbf{-topos } \mathbf{H}$$

$\rightarrow$

Higher moduli stacks of gauge fields

- a moduli stack of fields is **Fields** $\in$ **H**
- a **field configuration** on a $\begin{array}{c}\text{spacetime}\\ \text{worldvolume}\end{array}$ Σ is a map $\phi : \Sigma \to$ **Fields**;
- a *gauge transformation* is a homotopy $\kappa : \phi_1 \overset{\simeq}{\to} \phi_2 : \Sigma \to$ **Fields**
- a *higher gauge transformation* is a higher homotopy;
- the *BRST complex of gauge fields* on Σ is the infinitesimal approximation to the mapping stack $[\Sigma, \mathbf{Fields}]$.

Examples:

- for sigma-model field theory: **Fields** $= X$ is target space;
- for gauge field theory: **Fields** $= \mathbf{B}G_{\mathrm{conn}}$ is moduli stack of G-principal connections.
- in general both: σ-model fields and gauge fields are unified, for instance in "tensor multiplet" on super p-brane, Example 3 below

2)
Lagrangian field theory
Differential cohomology
Cohesion modality

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The action principle

For

- $\Sigma_{\mathrm{in}} \longrightarrow \Sigma \longleftarrow \Sigma_{\mathrm{out}}$ a cobordism (a *Feynman diagram*)
- $\mathbf{Fields}(\Sigma_{\mathrm{in}}) \xleftarrow{(-)|_{\Sigma_{\mathrm{in}}}} \mathbf{Fields}(\Sigma) \xrightarrow{(-)|_{\Sigma_{\mathrm{out}}}} \mathbf{Fields}(\Sigma_{\mathrm{out}})$ the space of *trajectories* of fields,

the *action functional* assigns a phase to each trajectory

$$\exp(iS_\Sigma) : \mathbf{Fields}(\Sigma) \to U(1)$$

and this is *Lagrangian* if there is *differential form* data $\mathbf{L} : \mathbf{Fields} \to \flat \mathbf{B}^n U(1)$ such that

$$\begin{tikzcd}
 & \mathbf{Fields}(\Sigma) \arrow[dl] \arrow[dr] & \\
\mathbf{Fields}(\Sigma_{\mathrm{in}}) \arrow[dr] & \arrow[l, phantom, "\overset{\exp(iS_\Sigma)=}{\underset{\exp(i\int_\Sigma \mathbf{L})}{\Longleftarrow}}"] & \mathbf{Fields}(\Sigma_{\mathrm{out}}) \arrow[dl] \\
 & \flat \mathbf{B} U(1) &
\end{tikzcd}$$

The need for differential cohesion

In order to formalize the action principle on gauge fields we hence need to

1. Characterize those $\left\{ \begin{array}{c} \infty\text{-toposes} \\ \text{homotopy type theories} \end{array} \right\}$ **H** whose $\left\{ \begin{array}{c} \text{objects} \\ \text{types} \end{array} \right\}$ may be interpreted as *differential geometric* spaces.
2. Axiomatize differential geometry and differential cohomology in such contexts.

$\rightarrow$ *differential cohesion*

The adjunction system defining differential cohesion

$$\mathbf{H} \underset{\Gamma}{\overset{\mathrm{LConst}}{\rightleftarrows}} \infty\mathrm{Grpd}$$

Every ∞-stack ∞-topos has an essentially unique global section geometric morphism to the base ∞-topos.

The adjunction system defining differential cohesion

$$\mathbf{H} \underset{\Gamma}{\overset{\mathrm{Disc}}{\rightleftarrows}} \infty\mathrm{Grpd}$$

Requiring the formation of locally constant ∞-stacks to be a full embedding means that we have a notion of *geometrically discrete objects* in $\mathbf{H}$.

The adjunction system defining differential cohesion

$$\begin{tikzcd}
\mathbf{H} \arrow[r, "\Gamma" description] & \infty\mathrm{Grpd} \arrow[l, hook', shift right=3, "\mathrm{Disc}"'] \arrow[l, hook, shift left=3, "\mathrm{coDisc}"]
\end{tikzcd}$$

Requiring the existence of an extra right adjoint means that we also have the inclusion of geometrically co-discrete (indiscrete) objects.

The adjunction system defining differential cohesion

$$\begin{tikzcd} \mathbf{H} \arrow[r, "\Gamma" description] & \infty\mathrm{Grpd} \arrow[l, hook', shift right=3ex, "\mathrm{Disc}" description] \arrow[l, hook, shift left=3ex, "\mathrm{coDisc}" description] \end{tikzcd}$$

Now Γ has the interpretation of sending a geometric homotopy type to its underlying ∞-groupoid of points, forgetting the geometric structure.

The adjunction system defining differential cohesion

$$\begin{tikzcd}[column sep=huge]
\mathbf{H} \arrow[r, shift left=4.5, "\Pi" description] \arrow[r, shift right=1.5, "\Gamma" description] & \infty\mathrm{Grpd} \arrow[l, hook', shift right=1.5, "\mathrm{Disc}" description] \arrow[l, hook', shift left=4.5, "\mathrm{coDisc}" description]
\end{tikzcd}$$

The crucial thing now is that for the ∞-topos $\mathbf{H}$ an extra left adjoint Π sends a geometric homotopy type to its *path* ∞*-groupoid* or *geometric realization*.

The adjunction system defining differential cohesion

$$\begin{tikzcd}
\mathbf{H} \arrow[r, shift left=4.5, "\Pi"] \arrow[r, shift right=1.5, "\Gamma"] & \infty\mathrm{Grpd} \arrow[l, hook', shift right=1.5, "\mathrm{Disc}"'] \arrow[l, hook', shift left=4.5, "\mathrm{coDisc}"]
\end{tikzcd}$$

If we further require that to preserve finite products then this means that the terminal object in **H** is geometrically indeed the point.

The adjunction system defining differential cohesion

$$\begin{tikzcd}[column sep=huge]
\mathbf{H} \arrow[r, shift left=4.5, "\Pi"] & \infty\mathrm{Grpd} \arrow[l, hook', shift right=1.5, "\mathrm{Disc}"'] \arrow[l, hook', shift left=4.5, "\mathrm{coDisc}"] \\
\end{tikzcd}$$

$$\mathbf{H} \underset{\mathrm{coDisc}}{\overset{\Gamma}{\rightleftarrows}} \infty\mathrm{Grpd}$$

If an adjoint quadruple of this form exists on $\mathbf{H}$ we say that $\mathbf{H}$ *is cohesive* or that its objects have the structure of *cohesively geometric homotopy types*.

The adjunction system defining differential cohesion

$$
\begin{tikzcd}[column sep=huge]
\mathbf{H}_{\mathrm{red}} \arrow[r, hook, shift right=1.5ex, "i_*"'] & \mathbf{H} \arrow[l, shift right=1.5ex, "i^*"'] \arrow[r, shift left=4.5ex, "\Pi"] \arrow[r, shift right=1.5ex, "\Gamma"'] & \infty\mathrm{Grpd} \arrow[l, hook', shift right=1.5ex, "\mathrm{Disc}"'] \arrow[l, hook, shift left=4.5ex, "\mathrm{coDisc}"]
\end{tikzcd}
$$

Consider moreover the inclusion of a cohesive sub-∞-topos $\mathbf{H}_{\mathrm{red}}$.

The adjunction system defining differential cohesion

$$
\begin{tikzcd}
\mathbf{H}_{\mathrm{red}} \arrow[r, hook, shift left=3, "i_!"] & \mathbf{H} \arrow[l, shift left=1, "i^*"] \arrow[r, shift left=3, "\Pi"] \arrow[r, shift right=1, "\Gamma"] & \infty\mathrm{Grpd} \arrow[l, hook', shift right=1, "\mathrm{Disc}"'] \arrow[l, hook', shift left=3, "\mathrm{coDisc}"] \\
\mathbf{H}_{\mathrm{red}} \arrow[r, hook, "i_*"] & \mathbf{H}
\end{tikzcd}
$$

If this has an extra left adjoint then this means that i^* is a projection map that contracts away from each object a geometric thickening *with no points*.

The adjunction system defining differential cohesion

$$\begin{tikzcd}
\mathbf{H}_{\mathrm{red}} \arrow[r, hook, shift left=3, "i_!"] \arrow[r, leftarrow, shift left=1, "i^*"] \arrow[r, hook, shift right=1, "i_*"] & \mathbf{H} \arrow[r, shift left=3, "\Pi"] \arrow[r, hookleftarrow, shift left=1, "\mathrm{Disc}"] \arrow[r, shift right=1, "\Gamma"] \arrow[r, hookleftarrow, shift right=3, "\mathrm{coDisc}"] & \infty\mathrm{Grpd}
\end{tikzcd}$$

This means that objects of $\mathbf{H}$ may have *infinitesimal thickening* ("formal neighbourhoods") and that $\mathbf{H}_{\mathrm{red}}$ is the full sub-∞-topos of the "reduced" objects: that have no infinitesimal thickening.

The adjunction system defining differential cohesion

$$\begin{tikzcd}[column sep=huge]
\mathbf{H}_{\mathrm{red}} \arrow[r, hook, shift left=6, "i_!"] \arrow[r, hook, shift right=2, "i_*"] \arrow[rr, bend left, shift left=10] & \mathbf{H} \arrow[l, shift right=2, "i^*"'] \arrow[l, shift left=6, "i^!"] \arrow[r, shift left=6, "\Pi"] \arrow[r, shift right=2, "\Gamma"] & \infty\mathrm{Grpd} \arrow[l, hook', shift right=2, "\mathrm{Disc}"'] \arrow[l, hook', shift left=6, "\mathrm{coDisc}"]
\end{tikzcd}$$

Finally that $\mathbf{H}_{\mathrm{red}}$ is itself cohesive means that $\Pi|_{\mathbf{H}_{\mathrm{red}}} = \Pi \circ i_!$ also preserves finite products.

From adjunctions to monads and modalities.

Such a system of two quadruple reflections on $\mathbf{H}$ is equivalently a system of two triple $\left\{ \begin{array}{c} \text{idempotent } \infty\text{-(co-)monads on} \\ \text{higher modalities in} \end{array} \right\}$ $\mathbf{H}$.

- $(\Pi \dashv \flat \dashv \sharp) : \;$
$$\begin{tikzcd} \mathbf{H} \arrow[r, shift left=2ex, "\Pi"] \arrow[r, shift right=2ex, "\Gamma"'] & \infty\mathrm{Grpd} \arrow[l, hook', "\mathrm{Disc}"'] \arrow[r, hook, shift left=2ex, "\mathrm{Disc}"] \arrow[r, hook, shift right=2ex, "\mathrm{coDisc}"'] & \mathbf{H} \arrow[l, "\Gamma"'] \end{tikzcd}$$

-
$$(\mathrm{Red} \dashv \Pi_{\mathrm{inf}} \dashv \flat_{\mathrm{inf}}) : \;
\begin{tikzcd} \mathbf{H} \arrow[r, shift left=2ex, "i^*"] \arrow[r, shift right=2ex, "i^!"'] & \mathbf{H}_{\mathrm{red}} \arrow[l, hook', "i_*"'] \arrow[r, hook, shift left=2ex, "i_!"] \arrow[r, hook, shift right=2ex, "i_*"'] & \mathbf{H} \arrow[l, "i^*"'] \end{tikzcd}$$

The modality system defining differential cohesion.

Π	shape modality	(idemp. ∞-monad)
$\perp$		
$\flat$	flat modality	(idemp. ∞-co-monad)
$\perp$		
$\sharp$	sharp modality	(idemp. ∞-monad)
Red	reduction modality	(idemp. ∞-co-monad)
$\perp$		
Π_{inf}	infinitesimal shape modality	(idemp. ∞-monad)
$\perp$		
$\flat_{\mathrm{inf}}$	infinitesimal flat modality	(idemp. ∞-co-monad)

The modality system defining differential cohesion.

Π	shape modality
$\perp$	
$\flat$	flat modality
$\perp$	
$\sharp$	sharp modality
Red	reduction modality
$\perp$	
Π_{inf}	infinitesimal shape modality
$\perp$	
$\flat_{\mathrm{inf}}$	infinitesimal flat modality

Models for differential cohesion

The following example accommodates most of contemporaty fundamental physics.

Theorem

Let $\mathrm{CartSp}_{\substack{\mathrm{synth} \\ \mathrm{super}}} := \left\{ \mathbb{R}^{p|q;k} = \mathbb{R}^p \times \mathbb{R}^{0|q} \times D^k \right\}_{p,q,k \in \mathbb{N}}$ *be the site of Cartesian formal supergeometric smooth manifolds with its standard open cover topology. The ∞-stack ∞-topos over it*

$$\mathrm{SynthDiffSuperSmooth}\infty\mathrm{Grpd} := \mathrm{Sh}_\infty(\mathrm{CartSp}_{\substack{\mathrm{synth} \\ \mathrm{super}}})$$

is differentially cohesive.

Objects are synthetic differential super-geometric smooth ∞-groupoids.

Remark

This is the homotopy-theoretic and super-geometric refinement of the traditional model for synthetic differential geometry known as the "Cahiers topos". [Dubuc 79].

References: Related work on differential cohesion

- The notion of differential cohesive ∞-toposes is a joint refinement to homotopy theory of W. Lawvere's
  - *synthetic differential geometry* [Lawvere 67, Dubuc 79]
  - *cohesion* [Lawvere 07]

  With hindsight one can see that the article *Some thoughts on the future of category theory* [Lawvere 91] is all about cohesion. What is called a "category of Being" there is a cohesive topos.
- Aspects of the infinitesimal modality triple $(\mathrm{Red} \dashv \Pi_{\mathrm{inf}} \dashv \flat_{\mathrm{inf}})$ appear
  - in [Simpson-Teleman 97] for the formulation of de Rham spacks;
  - in [Kontsevich-Rosenberg 04] for the axiomatization of formally étale maps.

3)
Local field theory
Higher category theory
Higher relations

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(...) [Fiorenza-Schreiber et al.] (...)

Observation

$$\mathrm{Bord}_n^{\mathrm{bry}} \longrightarrow \mathrm{Corr}_n(\mathbf{H}, \flat \mathbf{B}^n U(1))$$

$$\left(\;\color{red}{|}\!\!\longrightarrow * \;\right) \quad \mapsto \quad \begin{tikzcd}[column sep=small] & \mathbf{Fields}_{\color{red}{\partial}} \arrow[dl] \arrow[dr] & \\ * \arrow[dr, "0"'] & \arrow[l, phantom, "\swArrow"] & \mathbf{Fields} \arrow[dl, "\exp(iS)"] \\ & \flat \mathbf{B}^n U(1) & \end{tikzcd} .$$

By theorem 4.3.11 in [L09a].

References: Related work on local QFT by correspondences

- An early unfinished note is [Schreiber 08]
- For the special case of discrete higher gauge theory (∞-Dijkgraaf-Witten theory) a sketch of a theory is in section 3 and 8 of [FHLT].

4) Quantum field theory Motivic cohomology Abelianized relations

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Theorem (Nuiten-S.)

1. *On nice enough correspondences, forming twisted groupoid convolution algebras constitutes a functor*

$$\mathrm{Corr}_2^{\mathrm{nice}}(\mathrm{SmoothGrpd}, \mathbf{B}^2 U(1)) \xrightarrow{\int D\phi(-) := C^*(-)} \mathrm{KK}$$

   to KK-theory...

2. *...such that postcomposition with a prequantum boundary field theory*

$$\begin{tikzcd} \mathrm{Bord}_2^{\mathrm{bdr}} \arrow[r, "\exp(iS)"'] \arrow[rr, bend left, "\int D\phi\, \exp(iS(\phi))"] & \mathrm{Corr}_2(\mathrm{SmoothGrpd}, \mathbf{B}^2 U(1)) \arrow[r, "\int D\phi(-)"'] & \mathrm{KK} \end{tikzcd}$$

   subsumes the K-theoretic geometric quantization of Poisson manifolds – Example 1 below.

Outlook: Motivic quantization.

We may think of KK as a topological/differential geometric analog of *pure motives* [Connes-Consani-Marcolli 05]:

$$\mathrm{Mot}_2(\mathrm{SmoothGrpd}) := \mathrm{KK}\,.$$

Motives are abelianized correspondences.
We expect a general construction

$$\mathrm{Corr}_n(\mathbf{H}, \mathbf{B}^n U(1)) \xrightarrow[\text{“stabilize”}]{\int D\phi(-)} \mathrm{Mot}_n(\mathbf{H}) \;\; .$$

Then “motivic quantization” of local prequantum field theory:

$$\begin{tikzcd}
\mathrm{Bord}_n^{\mathrm{bdr}} \arrow[r, "\exp(iS)"'] \arrow[rr, bend left, "\int D\phi\, \exp(iS(\phi))"] & \mathrm{Corr}_n(\mathbf{H}, \mathbf{B}^n U(1)) \arrow[r, "\int D\phi(-)"'] & \mathrm{Mot}_n(\mathbf{H})
\end{tikzcd} \;\; .$$

References: Related work on motivic quantization

- Landsman: the natural target of quantization is KK-theory;
- Connes-Marcolli: KK-theory is the ncg-analog of motivic cohomology
- Baez-Dolan: from correspondences of finite groupoids to linear maps of finite vector spaces
- Lurie and FHLT: from correspondences of finite ∞-groupoids to maps of n-vector spaces.

Examples

Ex1 Holographic quantization of Poisson manifolds and D-branes.

Ex2 topol. ∞-YM $\overset{\text{bdr}}{\to}$ ∞-CS $\overset{\text{dfct}}{\to}$ ∞-WZW $\overset{\text{dfct}}{\to}$ ∞-Wilson surf.

Ex3 Super p-branes, e.g. M5 ($\overset{\text{event.}}{\rightsquigarrow}$ Khovanov, Langlands, ...)

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Example 1
Holographic quantization
of
Poisson manifolds and D-branes

(with J. Nuiten)

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Poisson manifolds

physics	mathematics
mechanical system	symplectic manifold (X, ω)
foliation by mechanical systems	Poisson manifold (X, π)
quantization of mechanical systems	quantization of Poisson manifolds

Observation: each Poisson manifold induces a 2-dimensional local Poisson-Chern-Simons theory whose mdouli stack of fields is the "symplectic groupoid" $\mathrm{SymGrp}(X, \pi)$ with local action functional

$$
\begin{array}{c}
\mathrm{SympGrpd}(X, \pi) \\
\Big\downarrow{\scriptstyle \exp(i S_{PCS})} \\
\mathbf{B}^2 U(1)_{\mathrm{conn}^1}
\end{array}
$$

The original Poisson manifold includes into the symplectic groupoid and naturally trivializes $\exp(iS_{PCS})$. So by Observation B it constitutes a canonical boundary condition for the 2-d Poisson-CS theory, exhibited by the correspondence

$$
\begin{tikzcd}
 & X \arrow[dl] \arrow[dr, "i"] & \\
{*} \arrow[dr] \arrow[rr, phantom, "\overset{\xi}{\swarrow}"] & & \mathrm{SymGrp}(X,\pi) \arrow[dl, "\chi"] \\
 & \mathbf{B}^2U(1) &
\end{tikzcd}
\simeq
\begin{tikzcd}
 & X \arrow[dl] \arrow[dd, "i^*\chi"] \arrow[dr, "i"] & \\
{*} \arrow[dr] \arrow[r, phantom, "\Downarrow\xi"] & \, & \mathrm{SymGrp}(X,\pi) \arrow[dl, "\exp(iS_{\mathrm{PCS}})"] \\
 & \mathbf{B}^2U(1) &
\end{tikzcd}
$$

Applying Theorem N, the groupoid convolution functor sends this to the co-correspondence of Hilbert bimodules

$$\mathbb{C} \xrightarrow{\Gamma(\xi)} C^*(X, i^*\chi) \xleftarrow{i^*} C^*(\mathrm{SymGrpd}, \chi) \ .$$

So if i is KK-orientable, then this boundary condition of the 2d PCS theory quantizes to the KK-morphism

$$\mathbb{C} \xrightarrow{\Gamma(\xi)} C^*(X, i^*\chi) \xrightarrow{i!} C^*(\mathrm{SymGrpd}, \chi)$$

hence to the class in twisted equivariant K-theory

$$i_![\xi] \in K(\mathrm{SympGrp}(X, \pi), \chi)\,.$$

The groupoid $\mathrm{SymGrp}(X, \pi)$ is a smooth model for the possibly degenerate space of symplectic leafs of (X, π) and this class may be thought of as the leaf-wise quantization of (X, π).

In particular when (X, π) is symplectic we have $\mathrm{SymGrpd}(X, \pi) \simeq *$ and $\xi = \mathbb{L}$ is an ordinary prequantum bundle and i is KK-oriented precisely if X is Spin^c. In this case

$$i_![\xi] = i_![\mathbb{L}] \in K(*) = \mathbb{Z}$$

is the traditional K-theoretic geometric quantization of (X, ω).

Similarly, for

$$\chi_B : X \to \mathbf{B}^2 U(1)$$

a B-field, a D-brane $i : Q \to X$ is a boundary condition given by

$$\begin{tikzcd}
 & Q \arrow[dl] \arrow[dr, "i"] & \\
* \arrow[dr] \arrow[rr, phantom, "\Swarrow \xi"] & & X \arrow[dl, "\chi"] \\
 & \mathbf{B}^2 U(1) &
\end{tikzcd}
\quad \simeq \quad
\begin{tikzcd}
 & Q \arrow[dl] \arrow[dr, "i"] \arrow[dd, "i^*\chi"] & \\
* \arrow[dr] \arrow[r, phantom, "\Swarrow \xi"] & {} & X \arrow[dl, "\chi"] \\
 & \mathbf{B}^2 U(1) &
\end{tikzcd}$$

where now ξ is the *Chan-Paton bundle* on the D-brane.

Proceeding as above shows that the quantization of this boundary condition in the 2d QFT which is the topological part of the 2d string σ-model gives the *D-brane charge*

$$i_![\xi] \in K(X, \chi) \, .$$

[Brodzki-Mathai-Rosenberg-Szabo 09]

In conclusion:

- The quantization of a Poisson manifold is equivalently its *brane charge* when regarded as a boundary condition of its 2d Poisson-Chern-Simons theory.

Conversely:

- The charge of a D-brane is equivalently the quantization of a particle on the brane charged under the Chan-Paton bundle.

References: Related work on quantization of Poisson manifolds

- Kontsevich + Cattaneo-Felder realize *perturbative* quantization of Poisson manifold holographically to perturbative quantization of 2d σ-model;
- [EH 06] completes Weinstein-Landsman program of geometric quantization of symplectic groupoids and obtains strict deformation quantization
- [Brodzki-Mathai-Rosenberg-Szabo 09] formalize D-brane charge in KK-theory

(...)

Example 2
∞-Chern-Simons
local prequantum field theory

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(...) [Fiorenza-Schreiber et al.] (...)

Example 3
Super p-branes

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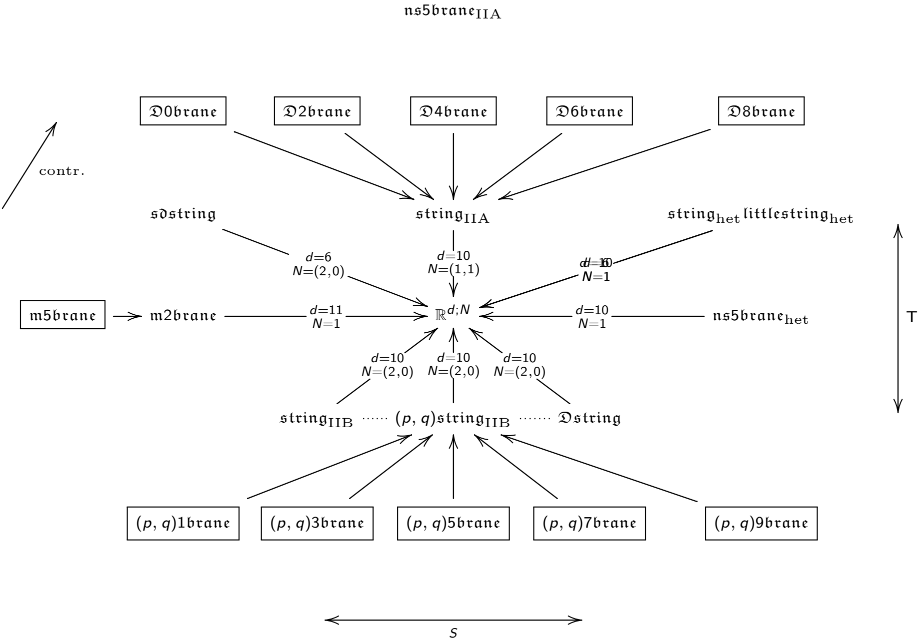
(...)
http://ncatlab.org/schreiber/show/infinity-Wess-Zumino-Witten+theory
http://ncatlab.org/schreiber/show/The+brane+bouquet
(...)



ns5brane_IIA
𝔇0brane
𝔇2brane
𝔇4brane
𝔇6brane
𝔇8brane
contr.
s𝔡string
string_IIA
string_het
littlestring_het
d=6 N=(2,0)
d=10 N=(1,1)
d=10 N=1
m5brane
m2brane
d=11 N=1
ℝ^{d;N}
d=10 N=1
ns5brane_het
T
d=10 N=(2,0)
d=10 N=(2,0)
d=10 N=(2,0)
string_IIB
(p, q)string_IIB
𝔇string
(p, q)1brane
(p, q)3brane
(p, q)5brane
(p, q)7brane
(p, q)9brane
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