Homotopy theory in type theory

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Review of type theory

- Type theory consists of rules for deriving typing judgments:

$$(x_1 \colon A_1), (x_2 \colon A_2), \ldots, (x_n \colon A_n) \vdash (b \colon B)$$

- The rules come in “packages” called type constructors.
- Each type constructor has four groups of rules: formation, introduction, elimination, and computation.
- Categorically: types are objects, terms are morphisms.
- Each type constructor corresponds to a categorical universal property.

Dependent eliminators

When we introduce predicates and dependent types, the eliminators of other types need to be generalized.

Example

- Suppose $(z\colon A + B) \vdash (P(z)\colon \mathsf{Type})$ is a predicate on $A + B$.
- We should be able to prove P by cases.
  1. Prove $(x\colon A) \vdash (p_A\colon P(\mathsf{inl}(x)))$.
  2. Prove $(y\colon B) \vdash (p_B\colon P(\mathsf{inr}(y)))$.
  3. Conclude $(z\colon A + B) \vdash (\mathsf{case}(z; p_A, p_B)\colon P(z))$.
- This looks like the "case split" eliminator for $A + B$, but the output type $P(z)$ depends on the element z that we are case-analyzing.

Therefore: we strengthen the elimination rules.

Dependent eliminators

Before

Suppose A, B, and C are types.

If $(x\colon A) \vdash (c_A : C)$ and $(y\colon B) \vdash (c_B : C)$, then for $p\colon A + B$ we have $\mathrm{case}(p, c_A, c_B) : C$.

After

Suppose A and B are types, and

$$(z\colon A + B) \vdash (C(z) : \mathrm{Type})$$

is a dependent type.

If $(x\colon A) \vdash (c_A : C(\mathrm{inl}(x)))$ and $(y\colon B) \vdash (c_B : C(\mathrm{inr}(y)))$, then for $p\colon A + B$ we have $\mathrm{case}(p, c_A, c_B) : C(p)$.

Dependent eliminators in categories

$$\begin{tikzcd}
 & & C \arrow[d] \\
A \arrow[urr] \arrow[rr] & & A+B \arrow[u, dashed, bend left] \\
 & B \arrow[uur] \arrow[ur] &
\end{tikzcd}$$

Dependent eliminators imply uniqueness

Theorem

Suppose $f, g \colon C^{A+B}$ *and that*

- *for all* $a \colon A$*, we have* $f(\mathrm{inl}(a)) = g(\mathrm{inl}(a))$*, and*
- *for all* $b \colon B$*, we have* $f(\mathrm{inr}(b)) = g(\mathrm{inr}(b))$*.*

Then for all $z \colon A + B$*, we have* $f(z) = g(z)$*.*

Proof.

Consider the dependent type

$$(z \colon A + B) \vdash (f(z) = g(z) : \mathrm{Type})$$

By the dependent eliminator for $A + B$, to construct a term of this type, it suffices to construct terms

$$\begin{aligned} (a \colon A) &\vdash (e_A : f(\mathrm{inl}(a)) = g(\mathrm{inl}(a))) \\ (b \colon B) &\vdash (e_B : f(\mathrm{inr}(b)) = g(\mathrm{inr}(b))) \end{aligned}$$

□

Function extensionality

It's more difficult to give a dependent eliminator for function types. Instead, we assert function extensionality directly as an axiom.

$$\Big(f, g \colon B^A\Big) \;\vdash\; \Big(\mathsf{funext} : \Big(\textstyle\prod_{x \colon A}(f(x) = g(x))\Big) \to (f = g)\Big)$$

Remarks

- Today I'll use both B^A and $A \to B$ for the function type.
- Later: more homotopical versions of both kinds of uniqueness.

Equality types

Equality types (or identity types) are a "positive type" (determined by the introduction rule):

1. For any type A and $a\colon A$ and $b\colon A$, there is a type $(a = b)$.
2. For any $a\colon A$, we have $\mathsf{refl}_a : (a = a)$.
3. Suppose $C(x, y, p)$ is a type dependent on three variables $x, y\colon A$ and $p\colon (x = y)$. Suppose moreover that for any $x\colon A$ we have an element $d(x) : C(x, x, \mathsf{refl}_x)$. Then for any x, y, p we have an element $J(d; x, y, p) : C(x, y, p)$.
4. $J(d; a, a, \mathsf{refl}_a)$ computes to $d(a)$.

Informally, ③ says

Elimination on equality

In order to do something with an arbitrary $p\colon (x = y)$,
it suffices to consider the case of $\mathsf{refl}_x : (x = x)$.

Equality is symmetric

Theorem

Suppose $p\colon (x = y)$*. Then* $p^{-1} : (y = x)$*.*

Proof.

By elimination, we may assume that p is $\mathsf{refl}_x : (x = x)$. But in this case, we can take p^{-1} to also be $\mathsf{refl}_x : (x = x)$. ☐

Just as in the cases of the dependent eliminator for coproducts, the desired conclusion $C(z)$ becomes $C(\mathsf{inl}(a))$ and $C(\mathsf{inr}(b))$, when we eliminate p the desired conclusion $(y = x)$ becomes $(x = x)$.

Equality is transitive

Theorem

Suppose $p\colon (x = y)$ *and* $q\colon (y = z)$*. Then* $p * q : (x = z)$*.*

Proof.

By elimination, we may assume that p is $\mathsf{refl}_x : (x = x)$. But in this case, we have $q\colon (x = z)$, so we can take $p * q$ to be just q. ☐

We could equally well have eliminated q, or both p and q.

Paths

We treat types as spaces/∞-groupoids/homotopy types, and we think of terms $p\colon (x = y)$ as paths $x \rightsquigarrow y$.

- Reflexivity becomes the constant path $\mathsf{refl}_x\colon x \rightsquigarrow x$.
- Transitivity becomes concatenation $x \overset{p*q}{\rightsquigarrow} z$ of $x \overset{p}{\rightsquigarrow} y \overset{q}{\rightsquigarrow} z$.
- Symmetry becomes reversal $y \overset{p^{-1}}{\rightsquigarrow} x$ of $x \overset{p}{\rightsquigarrow} y$.

But now there is more to say.

- Concatenation is associative: $\alpha_{p,q,r} : ((p * q) * r = p * (q * r))$.

2-paths

The “associator” $\alpha_{p,q,r}$ is coherent:

$$\begin{tikzcd}
 & (p * q) * (r * s) \arrow[ddr, squiggly] & \\
((p * q) * r) * s \arrow[ur, squiggly] \arrow[dd, squiggly] & & \\
 & & p * (q * (r * s)) \\
(p * (q * r)) * s \arrow[dr, squiggly] & & \\
 & p * ((q * r) * s) \arrow[uur, squiggly] &
\end{tikzcd}$$

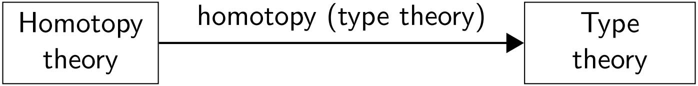
. . . or more precisely, there is a path between those two concatenations. . .

. . . which then has to be coherent. . .

∞-groupoids

Theorem (Lusmdaine,Garner–van den Berg)

The terms belonging to the iterated identity types of any type A form an ∞-groupoid.



Homotopy theory

homotopy (type theory)

Type theory

Note: Uses Batanin-Leinster ∞-groupoids (can also be done with simplicial versions).

Mapping on paths

Given $f\colon A \to B$, $x, y\colon A$, and a path $p\colon (x = y)$, we have an image path

$$\mathsf{map}(f, p) : (f(x) = f(y))$$

defined by eliminating on p:

- If p is refl_x, then $\mathsf{map}(f, p) :\equiv \mathsf{refl}_{f(x)}$.

Transporting along paths

Given $x, y \colon A$, $p \colon (x = y)$, and B dependent on A, we have the operation of transporting along p

$$\mathrm{trans}(p, -) : B(x) \to B(y).$$

defined by eliminating on p:

- If p is refl_x, then $\mathrm{trans}(p, -)$ is the identity map of $B(x)$.

Interpretation

We should view the map $B \to A$ as a fibration.
(In an $(\infty, 1)$-category, we can treat any map as a fibration.)

Paths for type constructors

For any type built using a type constructor, we can characterize its paths in terms of paths in its input types.

Example (Cartesian products)

- From $p\colon (a_1 = a_2)$ and $q\colon (b_1 = b_2)$, we can build

$$(p, q) : ((a_1, b_1) = (a_2, b_2))$$

- Given $z_1, z_2\colon A \times B$ and $r\colon (z_1 = z_2)$, we have

$$\mathsf{map}(\mathsf{fst}, r) : (\mathsf{fst}(z_1) = \mathsf{fst}(z_2))$$

$$\mathsf{map}(\mathsf{snd}, r) : (\mathsf{snd}(z_1) = \mathsf{snd}(z_2))$$

Paths in dependent sums

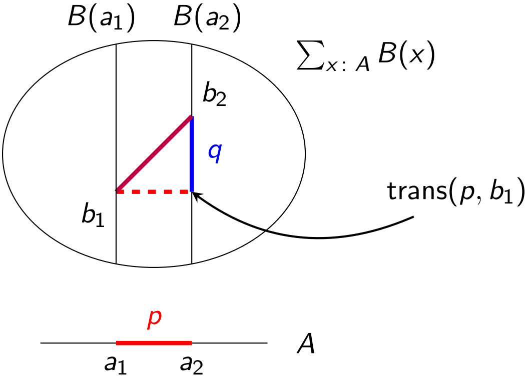
Suppose $a_1, a_2 \colon A$ and $b_1 \colon B(a_1)$ and $b_2 \colon B(a_2)$. A path

$$(a_1, b_1) = (a_2, b_2)$$

in $\sum_{x \colon A} B(x)$ should consist of

- A path $p \colon (a_1 = a_2)$ in A, and...
- what?
  - The expression $(b_1 = b_2)$ is ill-formed, since b_1 and b_2 have different types.
  - Instead we can use $q \colon (\mathsf{trans}(p, b_1) = b_2)$.

Paths in dependent sums



- In a fibration, we can lift the path p starting at b_1.
- We choose one lift and call its endpoint trans(p, b_1).
- Any other lift of p is determined by a path in the fiber $B(a_2)$.

Subsingletons in homotopy theory

Recall that logic is type theory restricted to subsingletons.

In homotopy type theory, we interpret "subsingleton" homotopically:

Theorem

For an object P in an $(\infty,1)$-category with products, TFAE:

1. *Each space* $\mathrm{Hom}(X,P)$ *is empty or contractible.*
2. *Any two morphisms* $X \rightrightarrows P$ *are homotopic.*
3. *The diagonal* $P \to P \times P$ *has a section.*
4. *The diagonal* $P \to P \times P$ *is an equivalence.*

h-Propositions

Definition

A type P is a proposition (or h-proposition or h-prop) if we have

$$(x\colon P), (y\colon P) \vdash (p : (x = y))$$

$$\begin{tikzcd}
P \arrow[d] \\
P \times P \arrow[u, dashed, bend left]
\end{tikzcd}$$

These are the “subsingletons” of homotopy type theory.

Building h-props

What ways do we have to obtain h-props?

- Most type constructors preserve h-props.
- For others ($+$ and $\sum$), we intend to apply "support".
- $(x = y)$ is not generally an h-prop, but has a support:
  - $(x = y)$ is the type of paths from x to y.
  - $\mathrm{supp}(x = y)$ is the assertion: there exists a path from x to y.
- For some types A, all equalities $(x = y)$ are h-props.
  - These are called sets or h-sets.
  - Certain types are always sets (e.g. $\mathbb{N}$, on Friday).
- But can we say anything homotopy-theoretic with this logic?

Internalizing h-props

How can we say in type theory “A is an h-prop”?

$$\mathrm{isProp}(A) :\equiv \mathrm{supp}\left(\prod_{x : A} \prod_{y : A} (x = y)\right) \qquad ?$$

Internalizing h-props

How can we say in type theory "A is an h-prop"?

$$\mathrm{isProp}(A) :\equiv \qquad \prod_{x\colon A} \prod_{y\colon A} (x = y) \qquad !$$

This is already an h-prop!

Theorem

For any A, we can construct a term in

$$\mathrm{isProp}(\mathrm{isProp}(A)).$$

Internalizing h-props

Theorem

For any A, isProp(isProp(A))*.*

Proof.

- Suppose $H, K \colon \mathrm{isProp}(A)$; we want $(H = K)$.
- By funext, suffices to show $H(x,y) = K(x,y)$ for all $x, y \colon A$.
- Now $\mathrm{map}(K(x,-), H(x,y))$ is a path in $\sum_z (x = z)$ from $K(x,x)$ to $K(x,y)$. In particular, it contains a path

$$\mathrm{trans}(H(x,y), K(x,x)) = K(x,y)$$

- Hence $H(x,y) * K(x,x) = K(x,y)$ (a fact).
- It suffices to prove $K(x,x) = \mathrm{refl}_x$.
  - The above argument with H being K, and y being x, yields $K(x,x) * K(x,x) = K(x,x)$.
  - Now cancel $K(x,x)$ (i.e. concatenate with $K(x,x)^{-1}$). ☐

Some subtleties

- We can loosely read $\prod_{x : A} \prod_{y : A} (x = y)$ as

  "for all $x, y : A$, we have a path $(x = y)$"

- But "for all $x, y : A$, there exists a path $(x = y)$" should be read to mean

  $$\prod_{x : A} \prod_{y : A} \mathsf{supp}(x = y)$$

  This asserts that "if A is nonempty, then it is connected."

- In $\prod_{x : A} \prod_{y : A} (x = y)$, the assigned path $(x = y)$ must depend continuously on x and y. This can be confusing until you get used to this meaning of "for all".

Some subtleties

- Type theory is a formal system.
- We can and do (and must, in practice) use informal language to speak and think about it.
- This depends on certain conventions about the formal interpretation given to informal words, which are sometimes subtly different to those used for some other formal system (like set theory).
- Fortunately, we have a computer proof assistant to type-check our proofs and guarantee that we didn't screw up!

Homotopy equivalences

Definition

A function $f\colon A \to B$ is a homotopy equivalence if there exists $g\colon B \to A$ and homotopies $g \circ f \sim \mathrm{id}_A$ and $f \circ g \sim \mathrm{id}_B$.

$$\mathrm{isHtpyEquiv}(f) :\equiv \mathrm{supp}\left(\sum_{g\colon B\to A}\Big((g \circ f = \mathrm{id}_A) \times (f \circ g = \mathrm{id}_B)\Big)\right)$$

This would not be an h-prop without supp. Can we avoid it?

Back to bijections

A function $f\colon A \to B$ between sets is a bijection if

1. There exists $g\colon B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.
2. OR: For each $b \in B$, the set $f^{-1}(b)$ is a singleton.
3. OR: There exists $g\colon B \to A$ such that $g \circ f = \mathrm{id}_A$ and also $h\colon B \to A$ such that $f \circ h = \mathrm{id}_B$.

Voevodsky equivalences

Definitions

The homotopy fiber of $f : A \to B$ at $b : B$ is

$$\mathrm{hfiber}(f, b) := \sum_{x : A} (f(x) = b).$$

A type X is contractible if it is an inhabited h-prop:

$$\mathrm{isContr}(X) := \mathrm{isProp}(X) \times X$$

Definition (Voevodsky)

f is an equivalence if each $\mathrm{hfiber}(f, b)$ is contractible:

$$\mathrm{isEquiv}(f) := \prod_{b : B} \mathrm{isContr}(\mathrm{hfiber}(f, b))$$

This is an h-prop.

H-isomorphisms

Definition (Joyal)

$f\colon A \to B$ is an h-isomorphism if we have $g\colon B \to A$ and a homotopy $g \circ f \sim \mathrm{id}_A$, and also $h\colon B \to A$ and a homotopy $f \circ h \sim \mathrm{id}_B$.

$$\mathrm{isHIso}(f) :\equiv \left(\sum_{g\colon B \to A} (g \circ f = \mathrm{id}_A)\right) \times \left(\sum_{h\colon B \to A} (f \circ h = \mathrm{id}_B)\right)$$

This is also an h-prop.

Adjoint equivalences

Given a homotopy equivalence, we can also ask for more coherence from $r\colon (g \circ f = \mathrm{id}_A)$ and $s\colon (f \circ g = \mathrm{id}_B)$.

(1a) For all $b\colon B$, we have $u(b)\colon (r(g(b)) = \mathrm{map}(g, s(b)))$.

(1b) For all $a\colon A$, we have $v(a)\colon (\mathrm{map}(f, r(a)) = s(f(a)))$.

(2a) For all $b\colon B$, we have $\ldots v(g(b) \ldots \mathrm{map}(g, u(b)) \ldots$

(2b) For all $a\colon A$, we have $\ldots u(f(a) \ldots \mathrm{map}(f, v(a)) \ldots$

$\vdots$

This gives an h-prop if we stop between any $(n\,\mathrm{a})$ and $(n\,\mathrm{b})$ (and then the rest can be constructed).

Definition

f is an adjoint equivalence if we have g, r, s, and u.

$$\mathrm{isAdjEquiv}(f) :\equiv \sum_{g\colon B \to A} \sum_{r\colon \ldots} \sum_{s\colon \ldots} \Big(r(g(b)) = \mathrm{map}(g, s(b)) \Big)$$

All equivalences are the same

Theorem

The following are equivalent:

1. *f is a homotopy equivalence.*
2. *f is a (Voevodsky) equivalence.*
3. *f is a (Joyal) h-isomorphism.*
4. *f is an adjoint equivalence.*

The last three are supp-free h-props, so we have equivalences

$$\mathsf{isEquiv}(f) \simeq \mathsf{isHIso}(f) \simeq \mathsf{isAdjEquiv}(f)$$

Definition

The type of equivalences between $A, B : \mathsf{Type}$ is

$$\mathsf{Equiv}(A, B) :\equiv \sum_{f : A \to B} \mathsf{isEquiv}(f).$$

The short five lemma

$$\begin{tikzcd}
\mathrm{hfiber}(f) \arrow[r] \arrow[d, "r"] & A \arrow[r, "f"] \arrow[d, "s"] & B \arrow[d, "t"] \\
\mathrm{hfiber}(g) \arrow[r] & C \arrow[r, "g"'] & D
\end{tikzcd}$$

Theorem

- *If s and t are equivalences, so is r.*
- *If r and t are equivalences, so is s.*

This is a theorem in type theory: A, B, C, D are types and we have a proof term

$$\big(p_1 : \mathrm{isEquiv}(s)\big),\ \big(p_2 : \mathrm{isEquiv}(t)\big)\ \vdash\ \big(q : \mathrm{isEquiv}(r)\big)$$

The 3 × 3 lemma

$$\begin{tikzcd}
 & \mathrm{hfiber}(h) \arrow[r, "r"] \arrow[d] & \mathrm{hfiber}(k) \arrow[d] \\
\mathrm{hfiber}(f) \arrow[r] \arrow[d, "s"'] & A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\
\mathrm{hfiber}(g) \arrow[r] & C \arrow[r, "g"'] & D
\end{tikzcd}$$

Theorem

There is an equivalence $\mathrm{hfiber}(r) \simeq \mathrm{hfiber}(s)$.

(Also a theorem in type theory.)

Homotopical uniqueness

Theorem

For any types A, B, C, *the map*

$$\lambda f.\big(\lambda a.f(\mathsf{inl}(a))\,,\,\lambda b.f(\mathsf{inr}(b))\big)\ :\ C^{A+B}\to C^A\times C^B$$

is an equivalence (using the dependent eliminator).

The type $C^{A+B}\to C^A\times C^B$ should be more consistently (but less legibly) written:

$$(C^A\times C^B)^{C^{A+B}}\quad\text{or}\quad\big((A+B)\to C\big)\to\big((A\to C)\times(B\to C)\big)$$

Awodey–Gambino–Sojakova have proven a much more general version of this, in the context we'll discuss on Friday.

Homotopical function extensionality

For $f, g \colon B^A$, there is a term

$$\mathsf{happly} \ : \ \left((f = g) \to \prod_{a \colon A} (f(a) = g(a)) \right)$$

defined by identity elimination:

$$\mathsf{happly}(\mathsf{refl}_f) :\equiv \lambda a.\mathsf{refl}_{f(a)}$$

Theorem (Voevodsky)

happly *is an equivalence (using the naive* funext*).*

Also works for dependent functions.

Paths in the universe

The only type whose path-types we have not determined (up to equivalence, in terms of other path-spaces) is the universe "Type".

$$\begin{tikzcd}
B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathsf{Type}} \arrow[d] \\
A \arrow[r] & \mathsf{Type}
\end{tikzcd}$$

If Type is the "classifying space" of types, then a path in Type should be an equivalence of types.

The univalence axiom

For $A, B \colon \mathsf{Type}$, we have

$$\mathsf{pathToEquiv}_{A,B} \;:\; \Big((A = B) \to \mathsf{Equiv}(A, B)\Big)$$

defined by identity elimination.
Note: $(A = B)$ is a path-type of "Type".

The Univalence Axiom (Voevodsky)

For all A, B, the function $\mathsf{pathToEquiv}_{A,B}$ is an equivalence.

$$\prod_{A \colon \mathsf{Type}} \; \prod_{B \colon \mathsf{Type}} \mathsf{isEquiv}(\mathsf{pathToEquiv}_{A,B})$$

In particular, every equivalence yields a path between types.

The meaning of univalence

The meaning of univalence

Given an equivalence $f\colon A \xrightarrow{\sim} B$, we can identify A with B along f.

In other words:

- When talking about A, B, and f, we "may as well assume" that B is A, and f is 1_A.
- Or: equivalent types can be treated as identical.

Proof.

Use the inverse of pathToEquiv, then the eliminator of equality. ☐

This is something we do informally all the time in mathematics. The univalence axiom gives it a precise formal expression.

The uses of univalence

1. The homotopy theory is nontrivial (Type is not an h-set).
2. (Voevodsky) Univalence implies funext.
3. For any type F, the type

   $$\sum_{A\colon \mathsf{Type}} \mathsf{supp}(A = F)$$

   is the classifying space for bundles with fiber F.
4. Computing homotopy groups! (on Friday)
5. Many more . . .