Categorical models of homotopy type theory

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Outline

1. Homotopy type theory in model categories
2. The universal Kan fibration
3. Models in $(\infty, 1)$-toposes

Homotopy type theory in higher categories

Recall:

homotopy type theory		$(\infty, 1)$-categories
$\times$, $+$ types	$\longleftrightarrow$	products, coproducts
equality types $(x = y)$	$\longleftrightarrow$	diagonals
$\prod$ types	$\longleftrightarrow$	local cartesian closure
univalent universe Type	$\longleftrightarrow$	object classifier

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These play different roles: type checking depends on computational equality.

- if a evaluates to b, and $c : C(a)$, then also $c : C(b)$.
  - In particular, if a evaluates to b, then $\mathsf{refl}_b : (a = b)$.
- if $p : (a = b)$ and $c : C(a)$, then only $\mathsf{transport}(p, c) : C(b)$.

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But computational equality is also stricter.

Example

Composition is computationally strictly associative.

$$g \circ f := \lambda x.g(f(x))$$

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- This is the sort of issue that homotopy theorists are intimately familiar with!
- We need a model for $(\infty, 1)$-categories with (at least) a strictly associative composition law.

Display map categories

Forget everything you know about homotopy theory; let's see how the type theorists come at it.

Definition

A display map category is a category with

- A terminal object.
- A subclass of its morphisms called the display maps, denoted $P \twoheadrightarrow A$ or $P \rightarrowtriangle A$.
- Any pullback of a display map exists and is a display map.

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- Any pullback of a display map exists and is a display map.

- A display map $P \twoheadrightarrow A$ is a type dependent on A.
- A display map $A \twoheadrightarrow 1$ is a plain type (dependent on nothing).
- Pullback is substitution.

Dependent sums of display maps

$$(x \colon A) \vdash (B(x) : \mathrm{Type})$$

If the types $B(x)$ are the fibers of $B \twoheadrightarrow A$, their dependent sum $\sum_{x \colon A} B(x)$ should be the object B.

$$(x \colon A) \vdash (B(x) : \mathrm{Type}) \qquad \begin{array}{c} B \\ \downarrow\!\!\!\!\downarrow \\ A \\ \downarrow\!\!\!\!\downarrow \\ 1 \end{array}$$

$$\vdash \left(\sum_{x \colon A} B(x) : \mathrm{Type} \right) \qquad \begin{array}{c} B \\ \downarrow\!\!\!\!\downarrow \\ 1 \end{array}$$

Dependent sums in context

More generally:

$$(x\colon A), (y\colon B(x)) \vdash (C(x,y) : \mathsf{Type}) \qquad \begin{array}{c} C \\ \downarrow\!\!\downarrow \\ B \\ \downarrow\!\!\downarrow \\ A \end{array}$$

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Dependent sums $\longleftrightarrow$ display maps compose

Aside: adjoints to pullback

- In a category $\mathscr{C}$, if pullbacks along $f\colon A \to B$ exist, then the functor
$$f^*\colon \mathscr{C}/B \longrightarrow \mathscr{C}/A$$
has a left adjoint Σ_f given by composition with f.

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- If f is a display map and display maps compose, then Σ_f restricts to a functor

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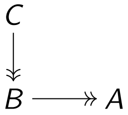
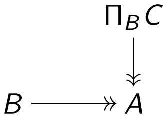
- A right adjoint to f^*, if one exists, is an "object of sections". $\mathscr{C}$ is locally cartesian closed iff all such right adjoints Π_f exist.

Dependent products of display maps

$$(x\colon A), (y\colon B(x)) \vdash (C(x,y) : \mathsf{Type})$$



$$(x\colon A) \vdash \left(\prod_{y\colon B(x)} C(x,y) : \mathsf{Type}\right)$$



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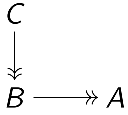

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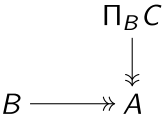


Dependent products $\longleftrightarrow$ “display maps exponentiate”

Identity types for display maps

The dependent identity type

$$(x\colon A), (y\colon A) \vdash ((x = y) : \mathsf{Type})$$

must be a display map

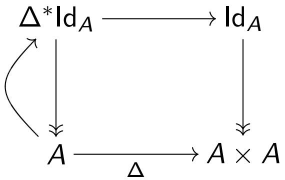
$$\begin{array}{c} \mathsf{Id}_A \\ \big\downarrow\!\!\!\!\downarrow \\ A \times A \end{array}$$

Identity types for display maps

The reflexivity constructor

$$(x\colon A) \ \vdash \ (\mathsf{refl}(x) : (x = x))$$

must be a section



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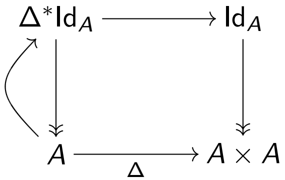
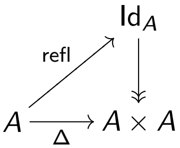
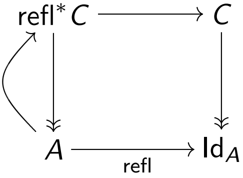


or equivalently a lifting



Identity types for display maps

The eliminator says given a dependent type with a section



there exists
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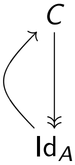

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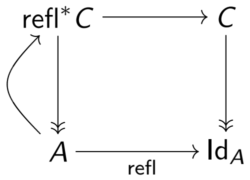

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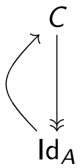
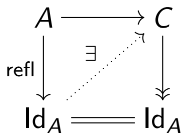


In other words, we have the lifting property



Identity types for display maps

In fact, refl has the left lifting property w.r.t. all display maps.

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle \mathsf{refl}}\big\downarrow & & \big\downarrow\!\!\!\!\downarrow \\ \mathsf{Id}_A & \xrightarrow[f]{} & B \end{array}$$

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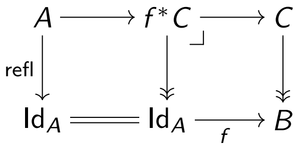

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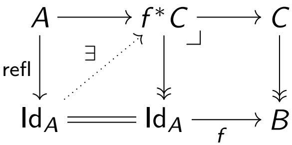

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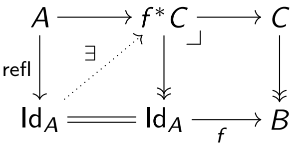


Conclusion

Identity types factor $\Delta\colon A \to A \times A$ as

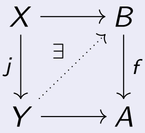
$$A \xrightarrow{\text{refl}} \mathsf{Id}_A \xrightarrow{q} A \times A$$

where q is a display map and refl lifts against all display maps.

Weak factorization systems

Definition

We say $j \boxslash f$ if any commutative square



admits a (non-unique) diagonal filler.

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$$\begin{array}{ccc} X & \longrightarrow & B \\ {\scriptstyle j}\downarrow & \exists \nearrow & \downarrow{\scriptstyle f} \\ Y & \longrightarrow & A \end{array}$$

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- $\mathcal{J}^{\boxslash} = \{\, f \mid j \boxslash f \quad \forall j \in \mathcal{J} \,\}$
- ${}^{\boxslash}\mathcal{F} = \{\, j \mid j \boxslash f \quad \forall f \in \mathcal{F} \,\}$

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Definition

A weak factorization system in a category is $(\mathcal{J}, \mathcal{F})$ such that

1. $\mathcal{J} = {}^{\boxslash}\mathcal{F}$ and $\mathcal{F} = \mathcal{J}^{\boxslash}$.
2. Every morphism factors as $f \circ j$ for some $f \in \mathcal{F}$ and $j \in \mathcal{J}$.

General factorizations

Theorem (Gambino–Garner)

In a display map category that models identity types, any morphism $g\colon A \to B$ factors as

$$A \xrightarrow{j} Ng \xrightarrow{f} B$$

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$$(y\colon B) \vdash Ng(y) :\equiv \mathsf{hfiber}(g, y) :\equiv \sum_{x\colon A} (g(x) = y)$$

is the type-theoretic mapping path space.

The identity type wfs

Corollary (Gambino-Garner)

In a type theory with identity types,

$$\left({}^{\boxslash}(\textit{display maps}), ({}^{\boxslash}(\textit{display maps}))^{\boxslash}\right)$$

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This behaves very much like (acyclic cofibrations, fibrations):

- Dependent types are like fibrations (recall "transport").
- Every map in ${}^{\boxslash}$(display maps) is an equivalence; in fact, the inclusion of a deformation retract.

Modeling identity types

Conversely:

Theorem (Awodey–Warren,Garner–van den Berg)

In a display map category, if

$$\left({}^{\boxslash}(\textit{display maps}), ({}^{\boxslash}(\textit{display maps}))^{\boxslash}\right)$$

is a "pullback-stable" weak factorization system, then the category (almost) models identity types.*

identity types $\longleftrightarrow$ weak factorization systems

Model categories

Definition (Quillen)

A model category is a category **C** with limits and colimits and three classes of maps:

- $\mathcal{C}$ = cofibrations
- $\mathcal{F}$ = fibrations
- $\mathcal{W}$ = weak equivalences

such that

1. $\mathcal{W}$ has the 2-out-of-3 property.
2. $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are weak factorization systems.

Type-theoretic model categories

Corollary

Let $\mathcal{M}$ be a model category such that

1. $\mathcal{M}$ (as a category) is locally cartesian closed.
2. $\mathcal{M}$ is right proper.
3. The cofibrations are the monomorphisms.

Then $\mathcal{M}$ (almost*) models type theory with dependent sums, dependent products, and identity types.

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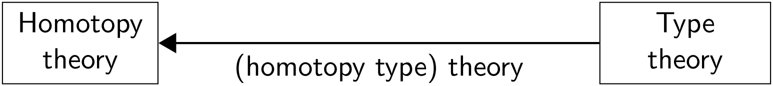

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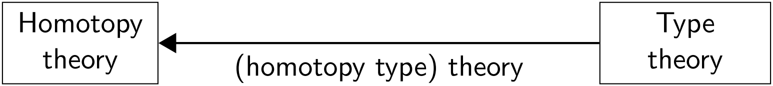

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- Simplicial sets with the Quillen model structure.

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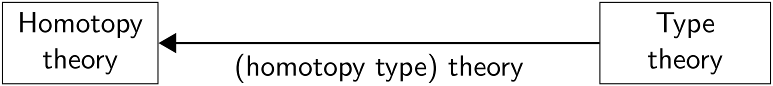

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- Any injective model structure on simplicial presheaves.

Homotopy type theory in categories

$$(x\colon A) \vdash p\colon \mathrm{isProp}(B(x))$$

$$\iff (x\colon A), (u\colon B(x)), (v\colon B(x)) \vdash (p_{u,v}\colon (u = v))$$

$\iff$ The path object $P_A B$ has a section in $\mathcal{M}/A$

$\iff$ Any two maps into B are homotopic over A

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$$(x\colon A) \vdash p\colon \mathsf{isContr}(B(x))$$
$$\iff (x\colon A) \vdash p\colon \mathsf{isProp}(B(x)) \times B(x)$$
$\iff$ Any two maps into B are homotopic over A and $B \twoheadrightarrow A$ has a section

$\iff$ $B \twoheadrightarrow A$ is an acyclic fibration

Homotopy type theory in categories

For $f\colon A \to B$,

$$\begin{aligned} \vdash\ p\colon \mathsf{isEquiv}(f) &\iff \vdash \prod_{y\colon B} \mathsf{isContr}(\mathsf{hfiber}(f,y)) \\ &\iff (y\colon B) \vdash \mathsf{isContr}(\mathsf{hfiber}(f,y)) \\ &\iff \mathsf{hfiber}(f) \twoheadrightarrow A \text{ is an acyclic fibration} \\ &\iff f \text{ is a (weak) equivalence} \end{aligned}$$

(Recall hfiber is the factorization $A \to Nf \twoheadrightarrow B$ of f.)

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Conclusion

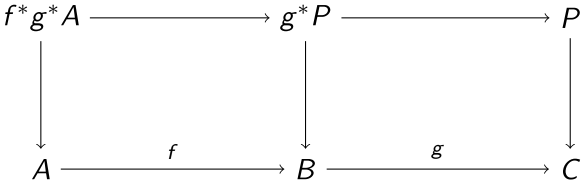
Any theorem about "equivalences" that we can prove in type theory yields a conclusion about weak equivalences in appropriate model categories.

Coherence

Another Problem

Type theory is even stricter than 1-categories!

Recall that substitution is pullback.



$a\colon A \vdash P(g(f(a)))$ $\quad$ $b\colon B \vdash P(g(b))$ $\quad$ $c\colon C \vdash P(c)$

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(g \circ f)^* A & \longrightarrow & P \\
\downarrow & & \downarrow \\
A & \xrightarrow{\;g \circ f\;} & C
\end{array}
$$

$a \colon A \vdash P((g \circ f)(a))$ $\qquad$ $c \colon C \vdash P(c)$

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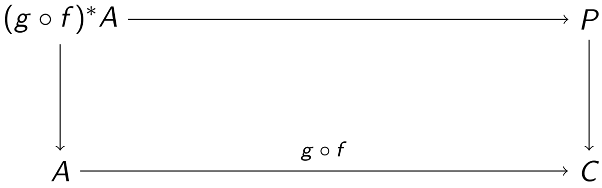

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Type theory is even stricter than 1-categories!

Recall that substitution is pullback.

$$\begin{array}{ccc} (g \circ f)^* A & \longrightarrow & P \\ \downarrow & & \downarrow \\ A & \xrightarrow{g \circ f} & C \end{array}$$

$a \colon A \vdash P(g(f(a)))$ $\qquad\qquad$ $c \colon C \vdash P(c)$

But, of course, $f^* g^* P$ is only isomorphic to $(g \circ f)^* P$.

Coherence with a universe

There are several resolutions; perhaps the cleanest is:

Solution (Voevodsky)

Represent dependent types by their classifying maps into a universe object.

Now substitution is composition, which is strictly associative (in our model category):

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{P} U$$

$$A \xrightarrow{\quad g \circ f \quad} C \xrightarrow{P} U$$

We needed a universe object anyway, to model the type Type and prove univalence.

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New problem

Need very strict models for universe objects.

Outline

Representing fibrations

(Following Kapulkin–Lumsdaine–Voevodsky)

Goal

A universe object in simplicial sets giving coherence and univalence.

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$$U_n \cong \mathrm{Hom}(\Delta^n, U) \simeq \{\text{fibrations over } \Delta^n\}$$

But $n \mapsto \{\text{fibrations over } \Delta^n\}$ is only a pseudofunctor; we need to rigidify it.

Well-ordered fibrations

A technical device (Voevodsky)

A well-ordered Kan fibration is a Kan fibration $p\colon E \to B$ together with, for every $x \in B_n$, a well-ordering on $p^{-1}(x) \subseteq E_n$.

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(with some size restriction, to make them sets).

The universal Kan fibration

Theorem

The forgetful map $\widetilde{U} \to U$ is a Kan fibration.

Proof.

A map $E \to B$ is a Kan fibration if and only if every pullback

$$\begin{array}{ccc} b^*E & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow \\ \Delta^n & \xrightarrow[b]{} & B \end{array}$$

is such, since the horns $\Lambda^n_k \hookrightarrow \Delta^n$ have codomain Δ^n. □

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Thus, of course, every pullback of $\widetilde{U} \to U$ is a Kan fibration.

The universal Kan fibration

Theorem

Every (small) Kan fibration $E \to B$ is some pullback of $\widetilde{U} \to U$:

$$\begin{array}{ccc} E & \longrightarrow & \widetilde{U} \\ \downarrow & \lrcorner & \downarrow \\ B & \longrightarrow & U \end{array}$$

Proof.

Choose a well-ordering on each fiber, and map $x \in B_n$ to the isomorphism class of the well-ordered fibration $b^*(E) \twoheadrightarrow \Delta^n$. $\square$

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It is essential that we have actual pullbacks here, not just homotopy pullbacks.

Type theory in the universe

Let the size-bound for U be inaccessible (a Grothendieck universe). Then small fibrations are closed under all categorical constructions.

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Let the size-bound for U be inaccessible (a Grothendieck universe). Then small fibrations are closed under all categorical constructions.

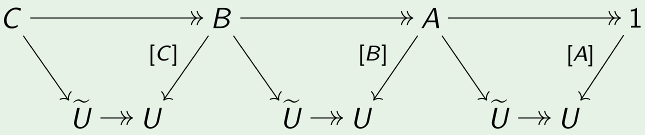
Now we can interpret type theory with coherence, using morphisms into U for dependent types.

Example

A context

$$(x \colon A), (y \colon B(x)), (z \colon C(x,y))$$

becomes a sequence of fibrations together with classifying maps:



in which each trapezoid is a pullback.

Strict cartesian products

Every type-theoretic operation can be done once over U, then implemented by composition.

Example (Cartesian product)

- Pull $\widetilde{U}$ back to $U \times U$ along the two projections π_1, π_2.

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- Their fiber product over $U \times U$ admits a classifying map:

$$\begin{array}{ccc} (\pi_1^* \widetilde{U}) \times_{U \times U} (\pi_2^* \widetilde{U}) & \longrightarrow & \widetilde{U} \\ \downarrow & \lrcorner & \downarrow \\ U \times U & \xrightarrow[{[\times]}]{} & U \end{array}$$

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U \times U & \xrightarrow[{[\times]}]{} & U
\end{array}
$$

- Define the product of $[A] \colon X \to U$ and $[B] \colon X \to U$ to be

$$X \xrightarrow{([A],[B])} U \times U \xrightarrow{[\times]} U$$

This has strict substitution.

Nested universes

Problem

So far the object U lives outside the type theory.
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So far the object U lives outside the type theory.
We want it inside, giving a universe type "Type" and univalence.

Solution

Let U' be a bigger universe. If U is U'-small and fibrant, then it has a classifying map:

$$\begin{array}{ccc} U & \longrightarrow & \widetilde{U}' \\ \big\downarrow & \lrcorner & \big\downarrow \\ 1 & \xrightarrow[u]{} & U' \end{array}$$

and the type theory defined using U' has a universe type u.

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Outline of proof.

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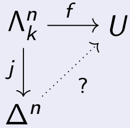

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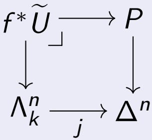

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Lemma

Any fibration $P \to \Lambda^n_k$ *is the pullback of some fibration over* Δ^n.

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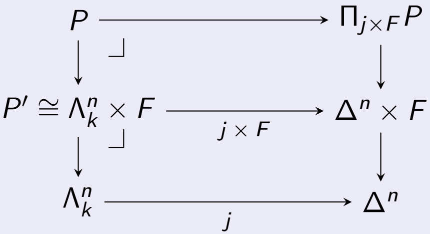

□

Univalence

We want to show that $PU \to \mathrm{Eq}(U)$ is an equivalence:

$$\begin{array}{ccc} PU & \xrightarrow{?} & \mathrm{Eq}(U) \\ & \searrow \quad \swarrow & \\ & U \times U & \end{array}$$

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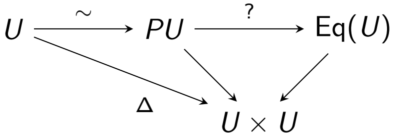

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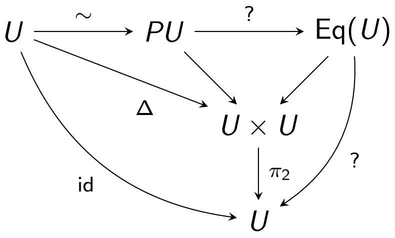

It suffices to show:

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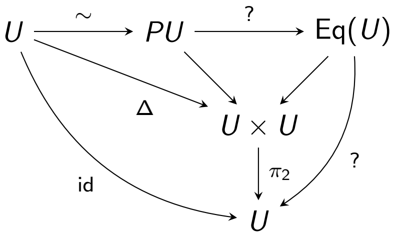


It suffices to show:

1. The composite $U \to \mathrm{Eq}(U)$ is an equivalence.
2. The projection $\mathrm{Eq}(U) \to U$ is an equivalence.
3. The projection $\mathrm{Eq}(U) \to U$ is an acyclic fibration.

Univalence

By representability, a commutative square

$$
\begin{array}{ccc}
\partial\Delta^n & \longrightarrow & \mathrm{Eq}(U) \\
{\scriptstyle i}\downarrow & & \downarrow \\
\Delta^n & \longrightarrow & U
\end{array}
$$

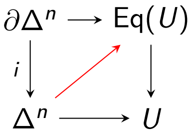
corresponds to a diagram

$$
\begin{array}{ccccc}
E_1 & & & & \\
 & \searrow & & & \\
\downarrow & & E_2 & \longrightarrow & \overline{E}_2 \\
 & \swarrow & \lrcorner & & \downarrow \\
\partial\Delta^n & & \xrightarrow[i]{} & & \Delta^n
\end{array}
$$

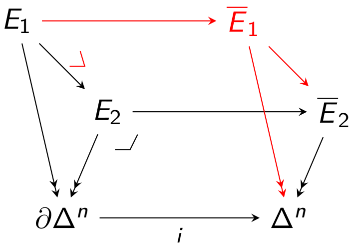
with $E_1 \to E_2$ an equivalence.

Univalence

By representability, a commutative square with a lift

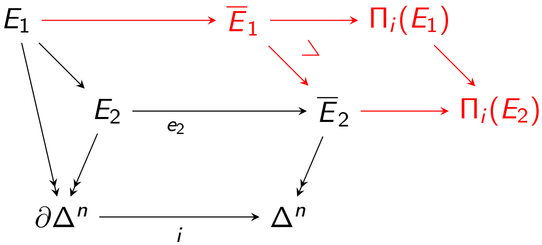


corresponds to a diagram



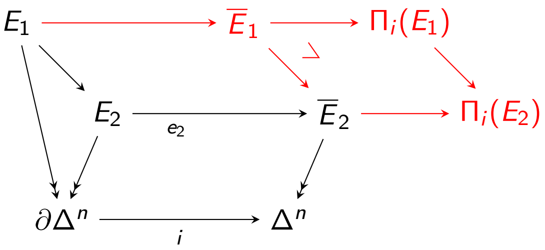
with $E_1 \to E_2$ and $\overline{E}_1 \to \overline{E}_2$ equivalences.

Univalence



E_1
$\overline{E}_1$
$\Pi_i(E_1)$
E_2
e_2
$\overline{E}_2$
$\Pi_i(E_2)$
$\partial\Delta^n$
i
Δ^n

Univalence



- By factorization, consider separately the cases when $E_1 \to E_2$ is (1) an acyclic fibration or (2) an acyclic cofibration.

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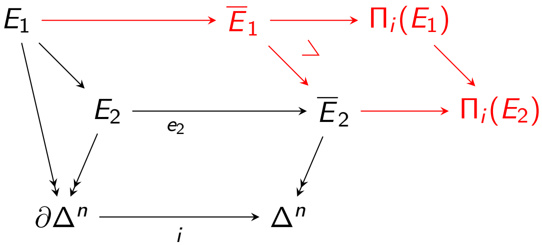

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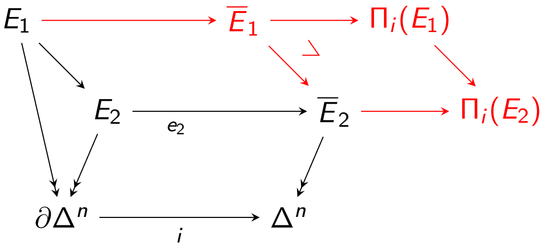

- By factorization, consider separately the cases when $E_1 \to E_2$ is (1) an acyclic fibration or (2) an acyclic cofibration.
- (1) $\overline{E}_1 \to \overline{E}_2$ is an acyclic fibration (Π_i preserves such).
- (2) $\overline{E}_1$ is a deformation retract of $\overline{E}_2$.

Outline

1. Homotopy type theory in model categories
2. The universal Kan fibration
3. Models in $(\infty, 1)$-toposes

$(\infty, 1)$-toposes

Definition

An $(\infty, 1)$-topos is an $(\infty, 1)$-category that is a left-exact localization of an $(\infty, 1)$-presheaf category.

Examples

- ∞-groupoids (plays the role of the 1-topos Set)
- Parametrized homotopy theory over any space X
- G-equivariant homotopy theory for any group G
- ∞-sheaves/stacks on any space
- "Smooth ∞-groupoids" (or "algebraic" etc.)

Univalence in categories

Definition (Rezk)

An object classifier in an $(\infty, 1)$-category $\mathcal{C}$ is a morphism $\widetilde{U} \to U$ such that pullback

$$\begin{array}{ccc} B & \longrightarrow & \widetilde{U} \\ \downarrow & \lrcorner & \downarrow \\ A & \longrightarrow & U \end{array}$$

induces an equivalence of ∞-groupoids

$$\mathrm{Hom}(A, U) \xrightarrow{\simeq} \mathrm{Core}(\mathcal{C}/A)_{\mathsf{small}}$$

("Core" is the maximal sub-∞-groupoid.)

$(\infty, 1)$-toposes

Theorem (Rezk)

An $(\infty, 1)$-category $\mathcal{C}$ is an $(\infty, 1)$-topos if and only if

1. *$\mathcal{C}$ is locally presentable.*
2. *$\mathcal{C}$ is locally cartesian closed.*
3. *κ-compact objects have object classifiers for $\kappa \gg 0$.*

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Corollary

If a combinatorial model category $\mathcal{M}$ interprets dependent type theory as before (i.e. it is locally cartesian closed, right proper, and the cofibrations are the monomorphisms), and contains universes for κ-compact objects that satisfy the univalence axiom, then the $(\infty, 1)$-category that it presents is an $(\infty, 1)$-topos.

$(\infty, 1)$-toposes

Conjecture

Every $(\infty, 1)$-topos can be presented by a model category which interprets dependent type theory with the univalence axiom.

Homotopy type theory is the internal logic of $(\infty, 1)$-toposes.

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Every $(\infty, 1)$-topos can be presented by a model category which interprets dependent type theory with the univalence axiom.

Homotopy type theory is the internal logic of $(\infty, 1)$-toposes.

If this is true, then anything we prove in homotopy type theory (which we can also verify with a computer) will automatically be true internally to any $(\infty, 1)$-topos. The "constructive core" of homotopy theory should be provable in this way, in a uniform way for "all homotopy theories".

Status of the conjecture

$\infty Gpd \dashrightarrow (\infty, 1)$-presheaves $\qquad (\infty, 1)$-toposes

Status of the conjecture

$$\infty Gpd \dashrightarrow (\infty, 1)\text{-presheaves} \dashrightarrow (\infty, 1)\text{-toposes}$$

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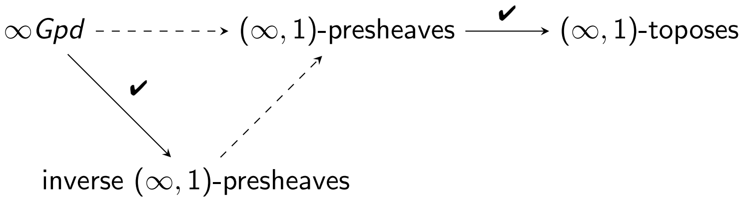

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