$*$-Autonomous Envelopes

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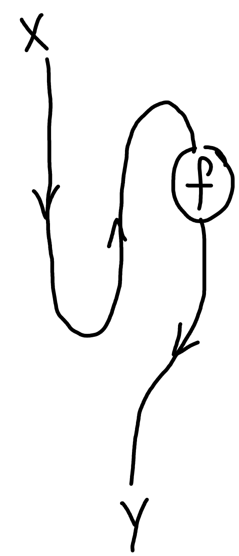
ACT@UCR Online Seminar, 22 Apr 2020

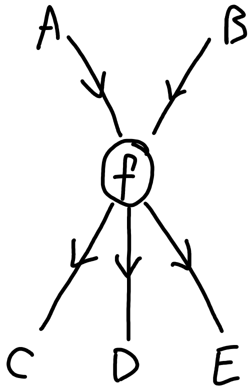
1. Review of string diagrams
2. Linear distributivity
3. The Chu construction
4. Polycategories
5. The Hyland envelope
6. Preserving tensors and cotensors



$$\begin{tikzcd} X \arrow[d, "1\otimes\ulcorner f\urcorner"] \\ X\otimes[X,Y] \arrow[d, "\mathrm{ev}"] \\ Y \end{tikzcd}$$

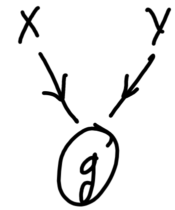
1. Review of String diagrams and duality

Let $\mathcal{C}$ be a symmetric monoidal category (smc).



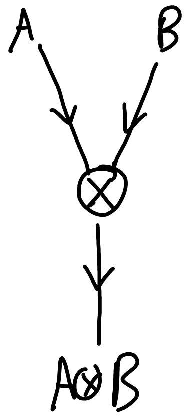
means

$$\begin{array}{c} A\otimes B \\ \downarrow f \\ C\otimes D\otimes E \end{array}$$

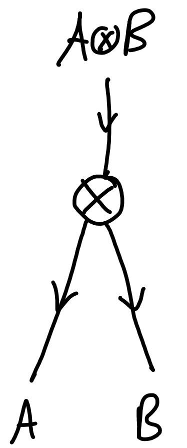


means

$$\begin{array}{c} X\otimes Y \\ \downarrow g \\ I \end{array}$$

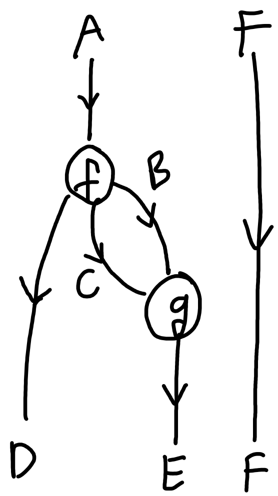


and



are both

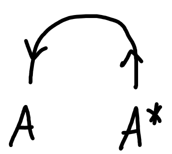
$$\begin{array}{c} A\otimes B \\ \downarrow 1_{A\otimes B} \\ A\otimes B \end{array}$$



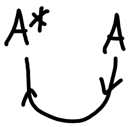
means

$$\begin{array}{c} A\otimes F \\ \downarrow f\otimes 1 \\ D\otimes C\otimes B\otimes F \\ \downarrow 1\otimes g\otimes 1 \\ D\otimes E\otimes F \end{array}$$

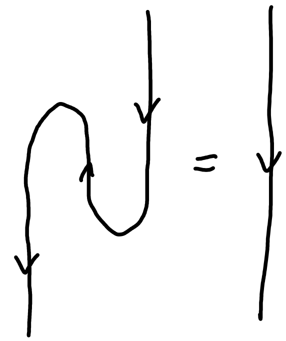
A dual of A is A^* with



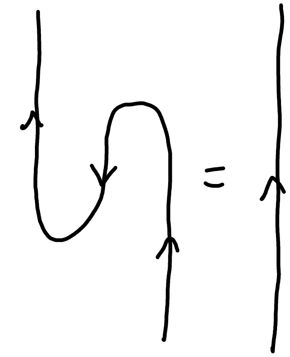
$= \quad I \xrightarrow{\eta} A\otimes A^*$ and



$= \quad A^*\otimes A \xrightarrow{\varepsilon} I$ such that

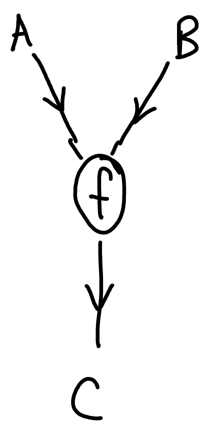


and

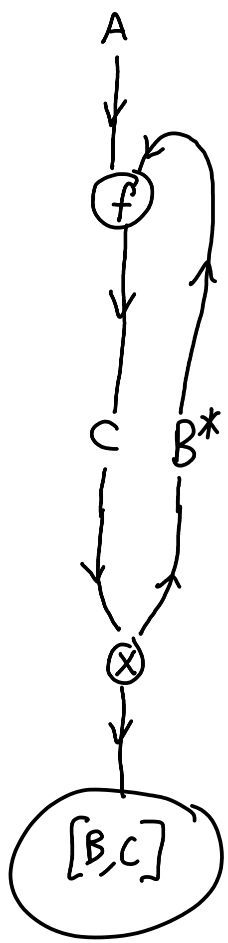


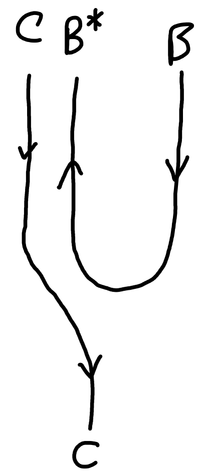
$\mathcal{C}$ is compact (= rigid) if every object has a dual.

A compact category is automatically closed with $[A,B] = A^*\otimes B$:

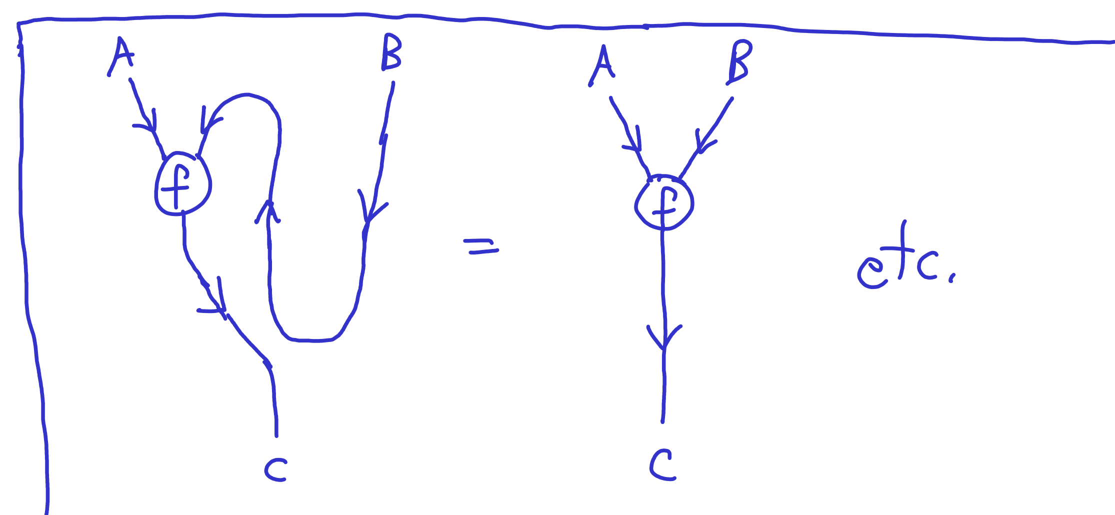


$\xmapsto{\text{curry}}$

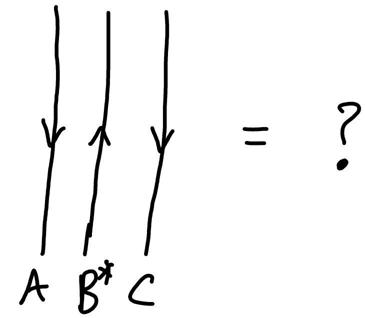




$= \quad [B,C]\otimes B \xrightarrow{ev} C$



What about closed symmetric monoidal categories (csmc) that are not compact? (e.g. Set, Ab, Vect, ...)

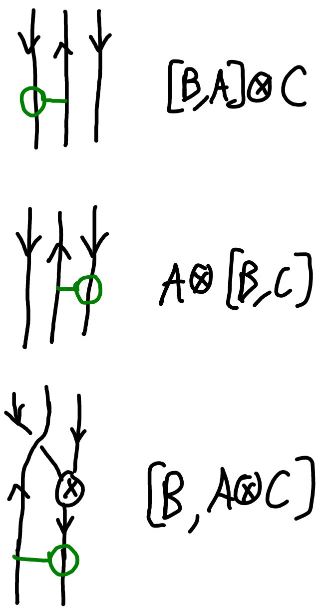


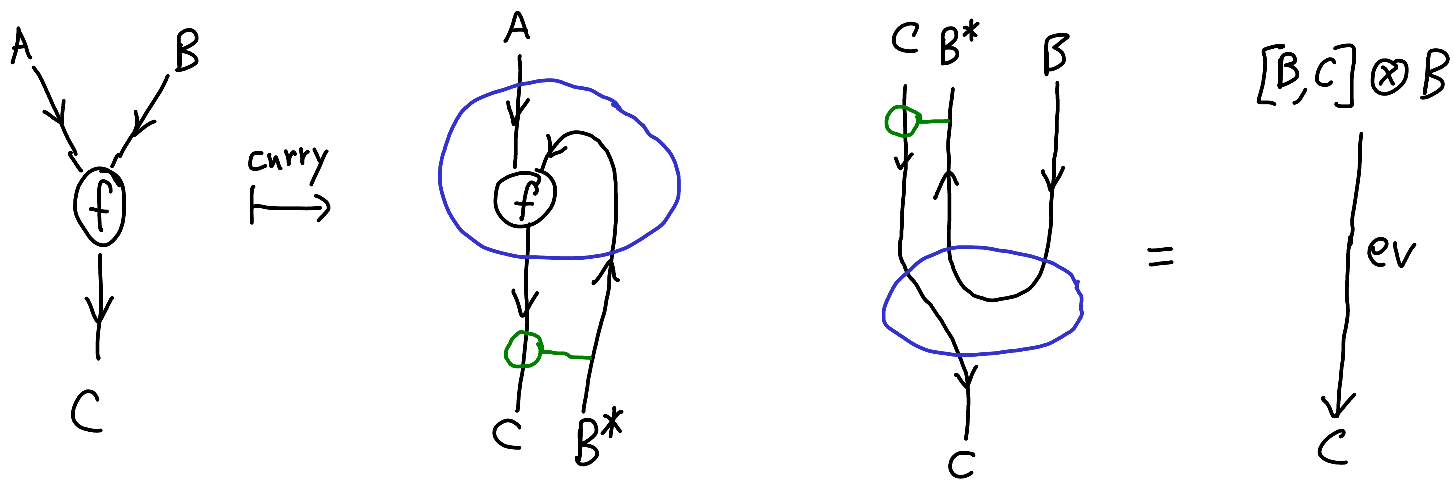
$[B,A]\otimes C$?

$A\otimes[B,C]$?

$[B, A\otimes C]$?

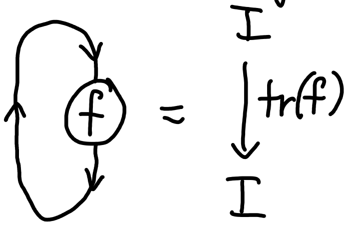
Baez-Stay (Rosetta stone) proposed clasps and bubbles:





The clasps carry the missing info; the bubbles prevent illegal operations.

The main "illegal operation" to worry about is the trace:

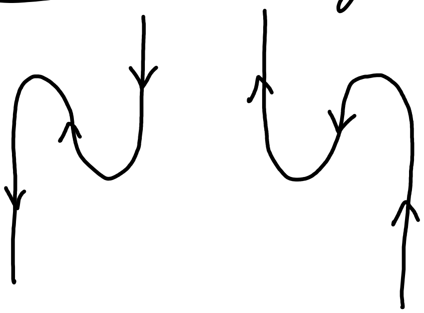


which exists in any compact category, but not in most csmc's.

Joyal-Street-Verity ("Traced monoidal categories"): A smc embeds in a compact one if and only if it is traced.

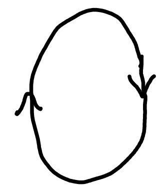
2. Linear Distributivity

OK:



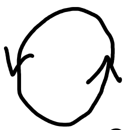
simply connected (no loops)

NOT OK:

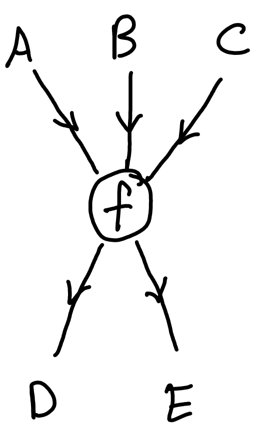


NOT simply connected !

What's the difference?

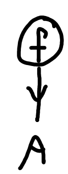
<u>Idea</u>: Forbid  by using a different tensor product for inputs and outputs, so the output of η and the input of ε don't match.

A <u>linearly distributive</u> category (née "weakly distributive", Cockett-Seely) has <u>two</u> symmetric monoidal structures $(\boxtimes, \top)$ and $(\oplus, \bot)$ plus structure and axioms.

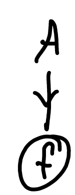


means

$$A \boxtimes B \boxtimes C \xrightarrow{f} D \oplus E$$

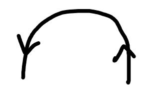


means $\top \to A$

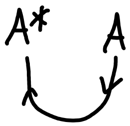


means $A \to \bot$

Let $\mathcal{D}$ be linearly distributive. A <u>dual</u> of A is A^* wit

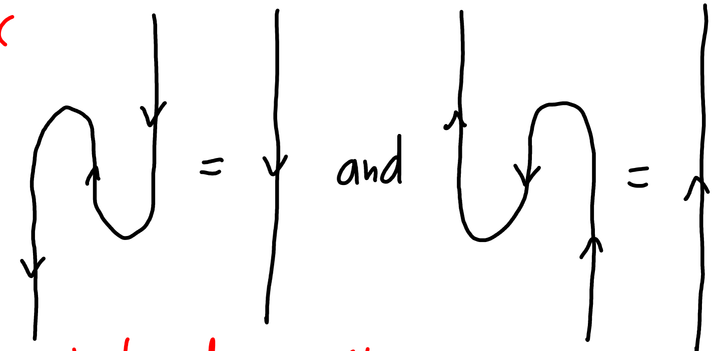


$= \quad \begin{array}{c} \top \\ \downarrow \eta \\ A \oplus A^* \end{array}$ and



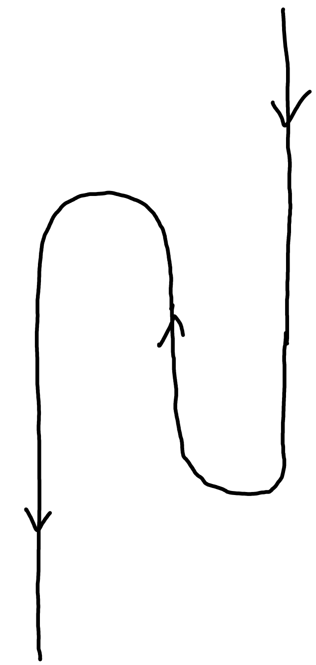
$= \quad \begin{array}{c} A^* \boxtimes A \\ \downarrow \varepsilon \\ \perp \end{array}$ such that

"



"

What does this mean?



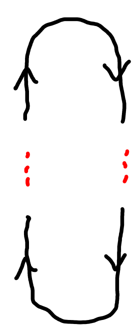
$$\begin{array}{c} A \\ \downarrow \cong \\ \top \boxtimes A \\ \downarrow \eta \boxtimes 1 \\ (A \oplus A^*) \boxtimes A \\ \downarrow \delta \\ A \oplus (A^* \boxtimes A) \\ \downarrow 1 \oplus \varepsilon \\ A \oplus \perp \\ \downarrow \cong \\ A \end{array}$$

The structure of $\mathcal{D}$ relating $\boxtimes$ and $\oplus$ is morphisms

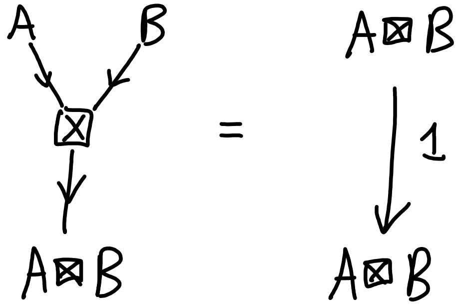
$$(A \oplus B) \boxtimes C \longrightarrow A \oplus (B \boxtimes C)$$

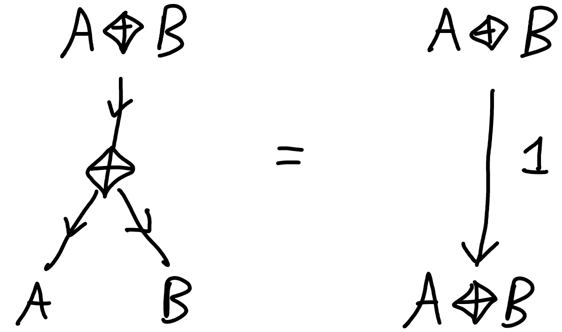
satisfying axioms.

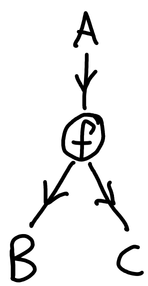
$\boxed{\text{NB}}$ Duals in a linearly distributive category do NOT have traces!

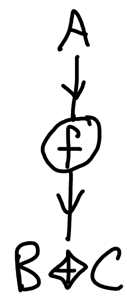


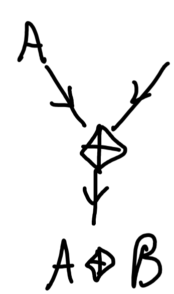
$$\begin{array}{c} \top \\ \downarrow \eta \\ A^* \oplus A \\ \neq \\ A^* \boxtimes A \\ \downarrow \varepsilon \\ \perp \end{array}$$

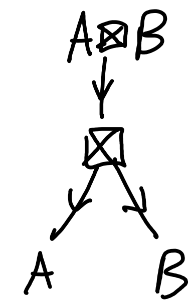
We have 

and 

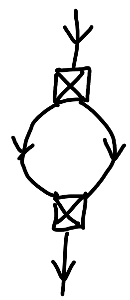
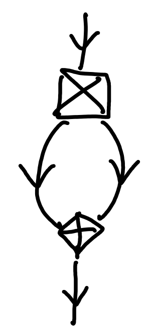
To make 

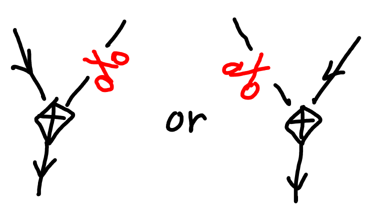
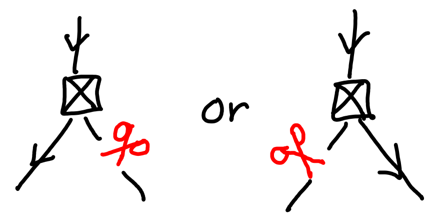
into 

we also allow 

and 

which are not morphisms in $\mathcal{D}$, but formal bits of string-diagram syntax.

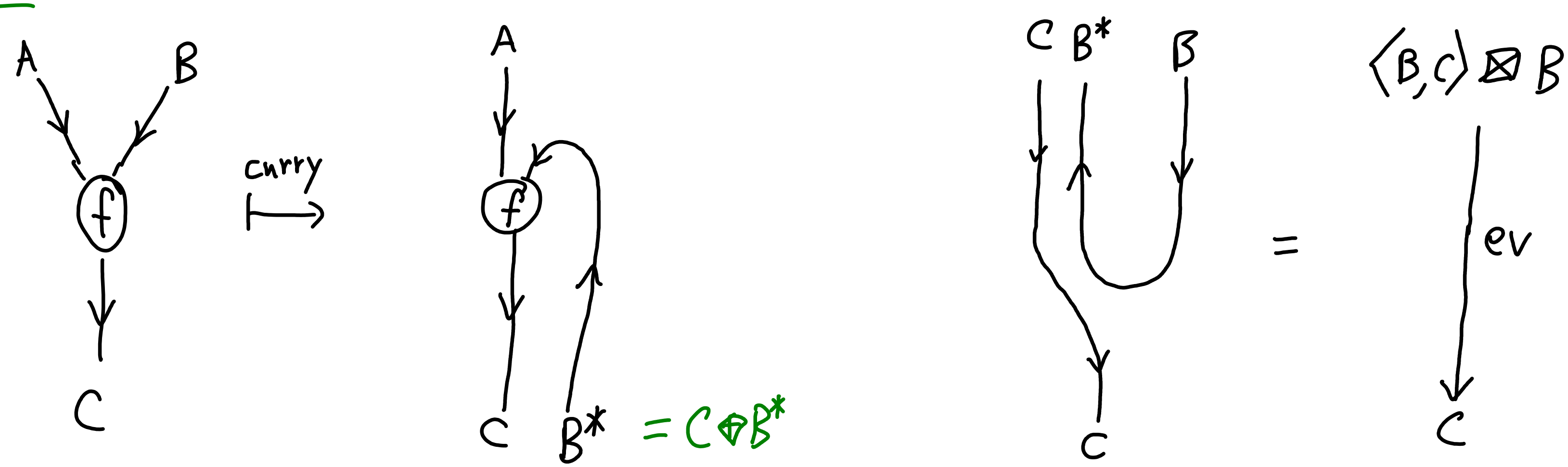
This complicates the validity criterion:  is OK, though not simply connected. But  is not OK.

Global proof net criterion (Girard, Danos-Regnier): A string diagram without $\top, \bot$ is valid iff it becomes simply-connected whenever we cut exactly one of the paired edges of each formal node:  , and 

(For $\top, \bot$ see Blute-Cockett-Seely-Trimble.)

A linearly distributive category with duals for all objects is called <u>$*$-autonomous</u>. (Can then recover $\oplus$ from duals by $A \oplus B = (A^* \boxtimes B^*)^*$, leading to the original definition by Barr as a csmc w/ duality $\mathcal{C} \simeq \mathcal{C}^{op}$.)

In a $*$-autonomous category, $\boxtimes$ is closed (and $\oplus$ is co-closed) with <u>$\langle A, B \rangle = A^* \oplus B$</u>. We can draw the same pictures <u>without</u> clasps/bubbles:



$$\langle B, C\rangle \boxtimes B = (C \oplus B^*) \boxtimes B \to C \oplus (B^* \boxtimes B) \xrightarrow{\varepsilon} C \oplus \bot \xrightarrow{\cong} C$$

Can we embed an arbitrary csmc in a $*$-autonomous category?
If so, we could use its string diagrams.

A more basic question: are there <u>any</u> $*$-autonomous categories?

1) Any smc is linearly distributive with $\boxtimes = \oplus = \otimes$. Then $*$-autonomous $\Leftrightarrow$ compact.

2) Any Boolean algebra is $*$-autonomous with $\boxtimes = \wedge$, $\oplus = \vee$, $A^* = \neg A$.
(But Boolean algebras don't categorify well.)

3. The Chu construction

How can we make a csmc $\mathcal{C}$ into a $*$-autonomous category $\mathcal{D}$?

Since we must have $\mathcal{D} \simeq \mathcal{D}^{op}$, one thing to try is $\mathcal{D} = \mathcal{C} \times \mathcal{C}^{op}$.

The simplest thing to try is to re-use the structure of $(\mathcal{C}, \otimes, I)$ for the first components of $(\boxtimes, \top)$:

$$(A^+, A^-) \boxtimes (B^+, B^-) = (A^+ \otimes B^+, \ ?)$$
$$\top = (I, \ ?)$$

(to be filled in later)

For the internal-hom $\langle \, , \rangle$ of $\boxtimes$ in $\mathcal{D}$ we must have bijections

$$\frac{\dfrac{\dfrac{\dfrac{\top \longrightarrow \langle (A^+, A^-), (B^+, B^-) \rangle}{(A^+, A^-) \longrightarrow (B^+, B^-)}}{A^+ \to B^+ \qquad B^- \to A^-}}{I \to [A^+, B^+] \qquad I \to [B^-, A^-]}}{I \longrightarrow [A^+, B^+] \times [B^-, A^-]}$$

$\xleftarrow{\text{becomes}}$ $I \longrightarrow [A^+, B^+] \times [B^-, A^-]$

$? \leftarrow ?$

so this certainly works if $\top = (I, 1)$.

so it's natural to guess

$$\langle (A^+, A^-), (B^+, B^-) \rangle = \left([A^+, B^+] \times [B^-, A^-], \ ? \right)$$

This determines everything since $\langle A,B\rangle = A^* \,\boxplus\, B = (A \boxtimes B^*)^*$!

$$(A^+,A^-) \boxtimes (B^+,B^-)^* = (A^+,A^-) \boxtimes (B^-,B^+)$$
$$= (A^+ \otimes B^-, \; ?)$$

$$\left((A^+,A^-) \boxtimes (B^+,B^-)^*\right)^* = (\,?\,, A^+ \otimes B^-) = \left([A^+,B^+] \times [B^-,A^-], \; ?\right)$$
$$= \left([A^+,B^+] \times [B^-,A^-], A^+ \otimes B^-\right)$$ and hence

$$(A^+,A^-) \boxtimes (B^+,B^-) = \left(A^+ \otimes B^+, \; [A^+,B^-] \times [B^+,A^-]\right).$$

<u>Theorem</u> Any csmc with finite products embeds by a strong sym. monoidal functor in a $*$-autonomous category $\mathrm{Chu}(\mathcal{C},1) = \mathcal{C} \times \mathcal{C}^{op}$, via $A \mapsto (A,1)$.

BUT this is not generally a <u>closed</u> embedding.

$$\langle (A,1),(B,1)\rangle = \left([A,B] \times [1,1], A \otimes 1\right) \cong \left([A,B], A \otimes 1\right) \not\cong \left([A,B], 1\right)$$

unless $A \otimes 1 \cong 1$ (e.g. if $1 = 0$).

More general Chu constructions: Choose $d \in \mathcal{C}$.

obj: $(A^+, A^-, A^+ \otimes A^- \to d)$

mor $\downarrow \uparrow$ such that a square commutes. $\rightsquigarrow$ $\mathrm{Chu}(\mathcal{C}, d)$.

$(B^+, B^-, B^+ \otimes B^- \to d)$

products become pullbacks

<u>E.g.</u> Lots of dualities embed in Chu's.

$\mathcal{C} = \mathrm{Vect}$, $d = \Bbbk$, $\mathrm{Hilb} \subset \mathrm{Chu}(\mathcal{C}, d)$

$V \otimes V \xrightarrow{\langle , \rangle} \Bbbk$.

$\mathcal{C} = \mathrm{Cat}$ $d = \mathrm{Set}$, $\mathrm{Adj} \subset \mathrm{Chu}(\mathcal{C}, d)$ (2-Chu constr.).

$(A^{op}, A, \hom : A^{op} \times A \to \mathrm{Set})$

$\mathcal{C} = \mathrm{Set}$ $d = 2$, Top as $(X, \mathcal{O}(X), \in)$.

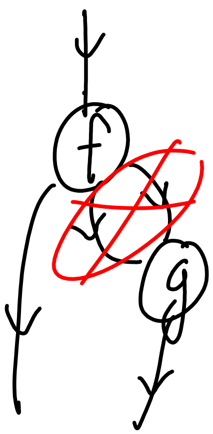
etc.... Stone duality, Pontryagin duality...

4. Polycategories.

multicategory $\mathcal{C}(\underbrace{A_1, \ldots, A_n}_{\text{domain}} ; B) \cong \mathcal{C}(A_1 \otimes \cdots \otimes A_n ; B)$.

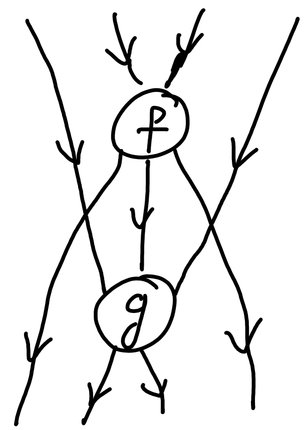
polycategory $\mathcal{C}(A_1, \ldots, A_m ; B_1, \ldots, B_n) \cong \mathcal{C}(A_1 \boxtimes \cdots \boxtimes A_m ; B_1 \oplus \cdots \oplus B_n)$

NOT a PROP — only allow composition along one object (simple connectivity)



ok in a PROP

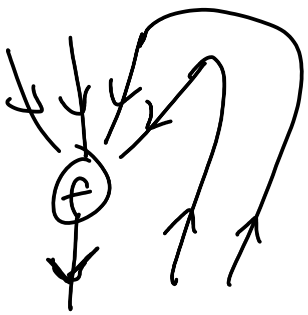
NOT in a polycat



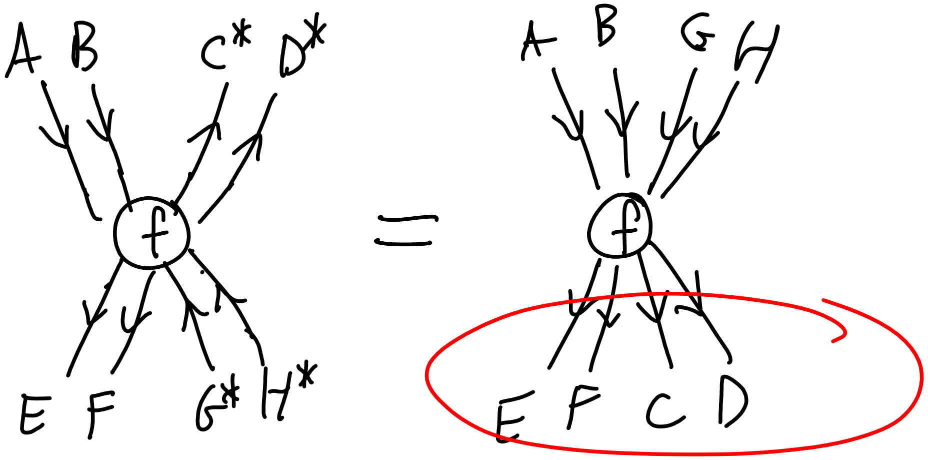
csmc $\mathcal{C}$ $\longrightarrow$ polycat $*\mathcal{C}$ $\longrightarrow$ csmc of modules $\xrightarrow{\text{Chu}}$ $*$-autonomous envelope.

formally add duals to $\mathcal{C}$: new objects A^*

$$*\mathcal{C}(A,B;C^*,D,E^*) = \mathcal{C}(A,B,C,E;D) = \mathcal{C}(A\otimes B\otimes C\otimes E;D)$$



$\boxed{\text{NB}}$ a lot of homsets are empty in $*\mathcal{C}$.



where on the right we mean morphisms in the underlying <u>multicategory</u> (not PROP) of $\mathcal{C}$. Thus, there are no such morphisms unless there is <u>exactly one</u> object in $\{E,F,C,D\}$.

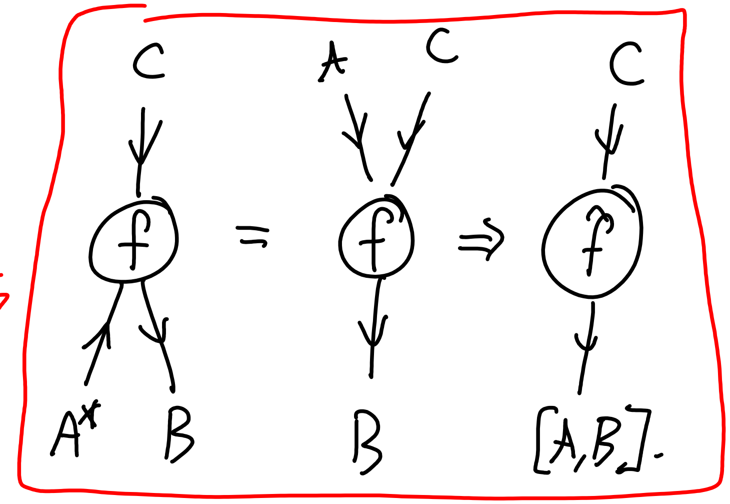
Can ${}^*\mathcal{C}$ have $\boxtimes$ and $\Diamond$? It has some, but not all.

$A \otimes B$ is $A \boxtimes B$ in ${}^*\mathcal{C}$.

$(A \otimes B)^*$ is $A^* \Diamond B^*$ in ${}^*\mathcal{C}$.

$[A, B]$ is $A^* \Diamond B$ in ${}^*\mathcal{C}$.

because



Need to add $\boxtimes$ and $\Diamond$ to ${}^*\mathcal{C}$, preserving the ones we had.

5. Hyland envelope.

$\mathcal{P}$ a polycat, $A \in \mathcal{P}$. Have two representables

$$\mathcal{L}_A(B_1 \cdots B_m ; C_1, \dots, C_n) = \mathcal{P}(B_1, \dots, B_m ; C_1, \dots, C_n, A).$$

$$_A\mathcal{L}(B_1 \dots B_m ; C_1 \dots C_n) = \mathcal{P}(B_1 \dots B_m, A ; C_1, \dots, C_n).$$

$\mathcal{L}_A, {}_A\mathcal{L} \in \mathrm{Mod}_{\mathcal{P}}$ (A module has hom-sets $M(B_1 \dots B_m ; C_1 \dots C_n)$ w/ actions by arrows in $\mathcal{P}$)

$\mathrm{Mod}_{\mathcal{P}}$ is a csmc, and has a canonical object "$\mathcal{P}$".

Analogy

$\mathcal{C} \hookrightarrow [\mathcal{C}^{op}, Set]$

preserves limits. ← bicomplete

consider only $X : \mathcal{C}^{op} \to Set$ preserving some colimits in $\mathcal{C}$

Thm (Hyland) $\mathcal{P} \xhookrightarrow[ff]{} \mathrm{Chu}(\mathrm{Mod}_{\mathcal{P}}, \mathcal{P}) = \mathrm{Env}_{\mathcal{P}}$.

(polycategorical Yoneda embedding).

Now consider only those modules that respect some $\mathcal{J}$ specified $\boxtimes$s and $\oplus$s in $\mathcal{P}$, to get an embedding that preserves those.

$$P \hookrightarrow \mathrm{Env}_{P,\mathcal{J}}.$$

$*$-autonomous

Answer: A csmc $\mathcal{C}$, $\quad \mathcal{C} \hookrightarrow \mathrm{Env}_{*\mathcal{C},\mathcal{J}}$

where $\mathcal{J}$ is
$$A \otimes B = A \boxtimes B$$
$$[A,B] = A^* \diamond B.$$

This is a closed symmetric monoidal embedding, so any string diagram we draw in the $*$-autonomous $\mathrm{Env}_{*\mathcal{C},\mathcal{J}}$ whose domain and codomain lie in (the image of $\mathcal{C}$) represents a unique morphism in $\mathcal{C}$, and the operations $\boxtimes$, $\top$, and $\langle , \rangle$ of $\mathrm{Env}_{*\mathcal{C},\mathcal{J}}$ restrict to the operations $\otimes$, I, and $[,]$ of $\mathcal{C}$.

But $\mathrm{Env}_{*\mathcal{C},\mathcal{J}}$ also has "new" objects like $A \diamond B$, for $A,B \in \mathcal{C}$.