Extraordinary multicategories

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The question

General question

In what type of category does a *(blank)* naturally live?

- A monoid in a multicategory is an object A with morphisms $() \xrightarrow{e} A$ and $(A, A) \xrightarrow{m} A$ satisfying axioms. (A monoid in a monoidal category is a special case.)
- An adjunction in a 2-category consists of morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$ with 2-morphisms $1_A \xrightarrow{\eta} gf$ and $fg \xrightarrow{\varepsilon} 1_B$ satisfying axioms.

Question A

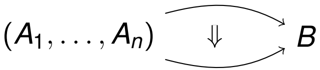
Where do monoidal categories naturally live?

2-multicategories

Definition

A 2-multicategory is a multicategory enriched over **Cat**.

Thus it has hom-categories $\mathscr{C}(A_1, \ldots, A_n\,;\ B)$, with composition functors as in a multicategory. We draw the morphisms in $\mathscr{C}(A_1, \ldots, A_n\,;\ B)$ as



Any monoidal 2-category (like **Cat**) has an underlying 2-multicategory, with

$$\mathscr{C}(A_1, \ldots, A_n\,;\ B) = \mathscr{C}(A_1 \otimes \cdots \otimes A_n\,,\ B)$$

Pseudomonoids

A pseudomonoid in a 2-multicategory is an object A with morphisms $() \xrightarrow{e} A$ and $(A, A) \xrightarrow{m} A$ and 2-isomorphisms like

$$\begin{array}{ccccc} & & (A,A) & & \\ & \overset{(m,1)}{\nearrow} & & \overset{m}{\searrow} & \\ (A,A,A) & & \Downarrow\cong & & A \\ & \underset{(1,m)}{\searrow} & & \underset{m}{\nearrow} & \\ & & (A,A) & & \end{array}$$

satisfying suitable axioms.

A pseudomonoid in **Cat** is precisely a monoidal category.

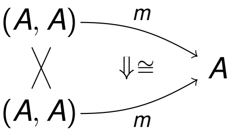
Symmetric pseudomonoids

Question B

Where do braided and symmetric pseudomonoids live?

We need to describe natural transformations with components such as $x \otimes y \to y \otimes x$, which switch the order of variables.

One answer: In a symmetric club, where a 2-morphism $f \xrightarrow{\alpha} g$ comes with a specified permutation relating the input strings of f and g. The symmetry is then a 2-isomorphism



(where the assigned permutation is the transposition in Σ_2).

Clubs

Definition (Kelly, 1972)

A (symmetric) club is a generalized multicategory for the “free symmetric strict monoidal category” monad on spans in **Cat**, whose category of objects is additionally discrete.

$$\mathscr{C}_0 \xleftarrow{\;t\;} \mathscr{C}_1 \xrightarrow{\;s\;} T\mathscr{C}_0$$

- $\mathscr{C}_0 =$ the discrete set of objects.
- $T\mathscr{C}_0 =$ category of finite lists of objects, with permutations.
- objects of $\mathscr{C}_1 =$ morphisms in $\mathscr{C}$.
- morphisms of $\mathscr{C}_1 =$ 2-morphisms in $\mathscr{C}$. The functor s assigns an underlying permutation to each 2-morphism.

More pseudomonoids

Question C

What about closed, $*$-autonomous, autonomous, pivotal, ribbon, and compact closed pseudomonoids?

These involve *contravariant* structure functors and *extraordinary natural* transformations. E.g. in a closed monoidal category, we have

- a functor $[-,-]\colon A^{op} \times A \to A$, and
- transformations

$$[x,y] \otimes x \to y$$
$$y \to [x, y \otimes x]$$

natural in y and extraordinary natural in x.

Extraordinary naturality

Definition (Eilenberg-Kelly, 1966)

Given $f\colon A^{op} \times A \times B \to C$ and $g\colon B \to C$, an extraordinary natural transformation $f \xrightarrow{\alpha} g$ consists of components $f(a,a,b) \xrightarrow{\alpha_{a,b}} g(b)$ such that

$$\begin{array}{ccc} f(a,a,b_1) & \longrightarrow & f(a,a,b_2) \\ {\scriptstyle \alpha_{a,b_1}}\downarrow & & \downarrow{\scriptstyle \alpha_{a,b_2}} \\ g(b_1) & \longrightarrow & g(b_2) \end{array} \qquad \begin{array}{ccc} f(a_2,a_1,b) & \longrightarrow & f(a_1,a_1,b) \\ \downarrow & & \downarrow{\scriptstyle \alpha_{a_1,b}} \\ f(a_2,a_2,b) & \xrightarrow[\alpha_{a_2,b}]{} & g(b) \end{array}$$

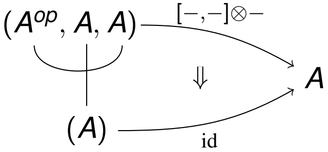
commute.

We can generalize to functors of higher arity.

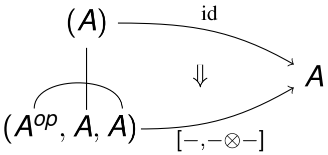
Graphs

Instead of a permutation, an extraordinary natural transformation is labeled by a graph matching up the input categories in pairs.

E.g. the transformations for a closed monoidal category are:



$$([x, y] \otimes x \to y)$$



$$(y \to [x, y \otimes x])$$

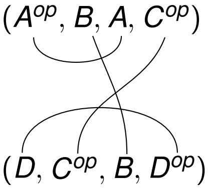
Clubs based on graphs

So we should consider T-multicategories

$$\mathscr{C}_0 \xleftarrow{\;t\;} \mathscr{C}_1 \xrightarrow{\;s\;} T\mathscr{C}_0$$

where

- objects of $T\mathscr{C}_0$ = finite lists of objects of $\mathscr{C}_0$ with assigned variances, e.g. $(A^{op}, B, A, C^{op}, D)$.
- morphisms of $T\mathscr{C}_0$ = graphs labeled by $\mathscr{C}_0$, e.g.



BUT. . .

Clubs based on graphs

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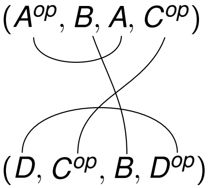
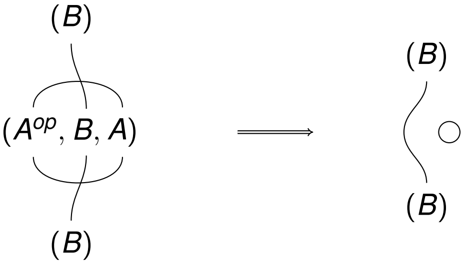


BUT…

The problem with loops

If we want to make $T\mathscr{C}_0$ into a category, we end up composing graphs in ways that create “loops.”



Theorem (Eilenberg-Kelly)

Extraordinary natural transformations can be composed if and only if the composite of their graphs produces no loops.

Dealing with loops

This is a problem! Ways that we might deal with it:

- Compose graphs by throwing away any loops that appear (Kelly's choice).
  - **Pros:** We can describe the free club on a closed pseudomonoid, and the monad it generates.
  - **Cons:** We can't do the same for compact closed pseudomonoids. Also, **Cat** is not itself a club of this sort.
- Don't let ourselves compose graphs that create loops.
- Generalize the notion of extraordinary natural transformation in a way that can deal with loops.

Extraordinary multicategories

Let **PCat** denote the category of partial categories: directed graphs with a partially defined composition operation, with identities, which is associative insofar as it is defined.

Now we can define a monad $T_{\mathbf{eo}}$ on **PCat** with

- objects of $T_{\mathbf{eo}}\mathscr{C}$ = finite lists of objects of $\mathscr{C}$ with variances,
- morphisms of $T_{\mathbf{eo}}\mathscr{C}$ = loop-free graphs labeled by $\mathscr{C}$, with composition defined whenever no loops occur.

Definition

An extraordinary 2-multicategory is a $T_{\mathbf{eo}}$-multicategory where $\mathscr{C}_0$ is discrete and $\mathscr{C}_1 \xrightarrow{s} T_{\mathbf{eo}}\mathscr{C}_0$ reflects composability.

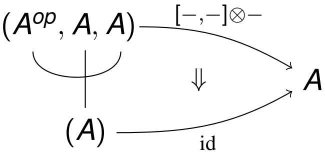
(The last condition implements the Eilenberg-Kelly theorem.)

Back to pseudomonoids

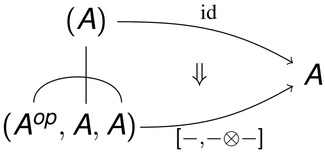
A closed pseudomonoid in an extraordinary 2-multicategory is a pseudomonoid together with a morphism

$$[-,-]\colon (A^{op}, A) \to A$$

and extraordinary 2-morphisms



and



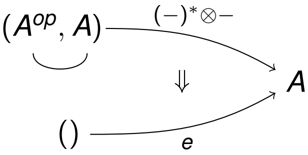
satisfying suitable axioms.

Back to pseudomonoids

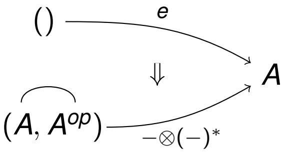
A compact closed pseudomonoid in an extraordinary 2-multicategory is a symmetric pseudomonoid together with a morphism

$$(-)^*: (A^{op}) \to A$$

and extraordinary 2-morphisms



and



satisfying suitable axioms.

Profunctors

Definition

Let $f\colon A \to C$ and $g\colon B \to C$ be functors, and $H\colon B \nrightarrow A$ be a profunctor, i.e. a functor $A^{op} \times B \to \mathbf{Set}$. An *$H$-natural transformation* $f \to g$ is a natural transformation $H(a,b) \to \hom_C(f(a), g(b))$.

- If $A = B$ and $H = \hom_A$, an H-natural transformation is an ordinary natural transformation.
- If $f\colon D^{op} \times D \times B \to C$ and $g\colon B \to C$, and

  $$H\big((d_1, d_2, b_1),\ b_2\big) = \hom_D(d_2, d_1) \times \hom_B(b_1, b_2),$$

  an H-natural transformation is an extraordinary natural transformation of graph " $^{\cup}\!\!/$ ".

But we can *always* compose an H-natural transformation with a K-natural one to get a $(K \circ H)$-natural one.

Compact closed double categories

A double category has objects, arrows (drawn horizontally), proarrows (drawn vertically), and square-shaped 2-cells.

$$\begin{array}{ccc} A & \longrightarrow & C \\ \uparrow\!\!\bullet & \Downarrow & \uparrow\!\!\bullet \\ B & \longrightarrow & D \end{array}$$

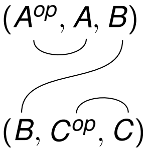
It is compact closed if it is symmetric monoidal, and every object A has a dual A^{op} with a unit and counit that are *proarrows*.

Example

In **Cat**, the arrows are functors and the proarrows are profunctors. The dual of A is A^{op}, and the unit and counit are both $\hom_A\colon A^{op} \times A \to \mathbf{Set}$, regarded either as a profunctor $1 \nrightarrow A \times A^{op}$ or $A^{op} \times A \nrightarrow 1$.

Extraordinary naturality, again

In a compact closed double category, any labeled graph can be composed into a proarrow. E.g.:



$$\Longrightarrow \qquad \begin{array}{c} A^{op} \otimes A \otimes B \\ \uparrow \bullet\ \eta_A \otimes 1_B \otimes \varepsilon_C \\ B \otimes C^{op} \otimes C \end{array}$$

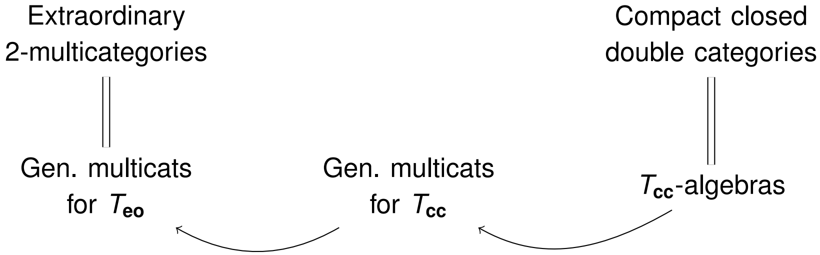
Thus an extraordinary 2-morphism labeled by such a graph can be defined to mean a square

$$\begin{array}{ccc} A^{op} \otimes A \otimes B & \xrightarrow{f} & D \\ \eta_A \otimes 1_B \otimes \varepsilon_C \ \bullet\!\uparrow & \Downarrow & \uparrow\!\bullet\ 1_D \\ B \otimes C^{op} \otimes C & \xrightarrow[g]{} & D \end{array}$$

We can again define (compact) closed pseudomonoids and so on.

The connection

The monad $T_{\mathbf{cc}}$ on double categories, whose algebras are compact closed double categories, is also essentially built out of graphs (with loops). Thus we (sort of) have a morphism of monads $T_{\mathbf{cc}} \to T_{\mathbf{eo}}$, giving rise to forgetful functors:



The composite of these functors is exactly the construction we just described.

Further References and Remarks

- Double categories of profunctors also tend to have *companions* and *conjoints* for arrows, making them into *proarrow equipments* (Wood, 1982).
- If we discard the arrows and consider only the proarrows, we obtain the functorial calculus of autonomous bicategories (Street, 2003). The double category version exists in unpublished work of Baez and Mellies.
- The graphs labeling extraordinary natural transformations can also be identified with a 2-dimensional shadow of the *surface diagrams* for autonomous bicategories (and double categories). But making that precise is difficult. . .

Graphs vs. surface diagrams