Geometric HoTT and comonadic modalities

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Geometry in Modal Homotopy Type Theory workshop
Carnegie Mellon University
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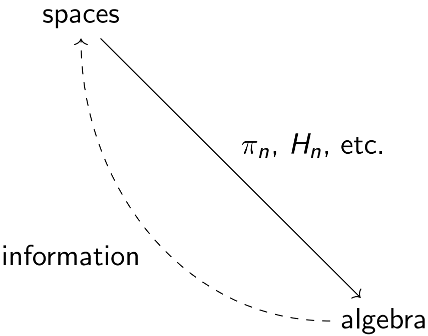
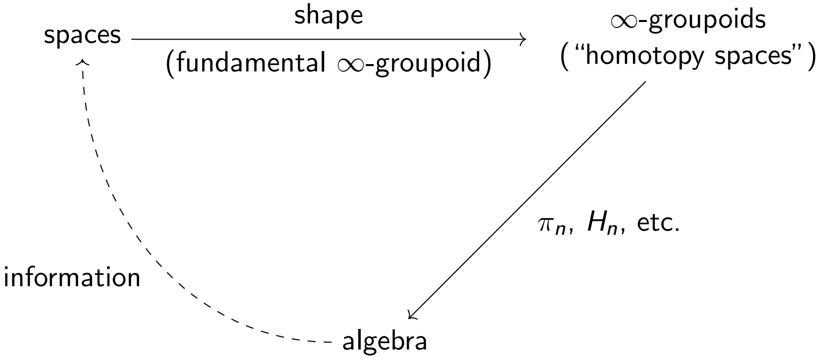
Outline

1. Cohesion
2. Geometric type theory
3. Geometric modalities
4. Modal type theory

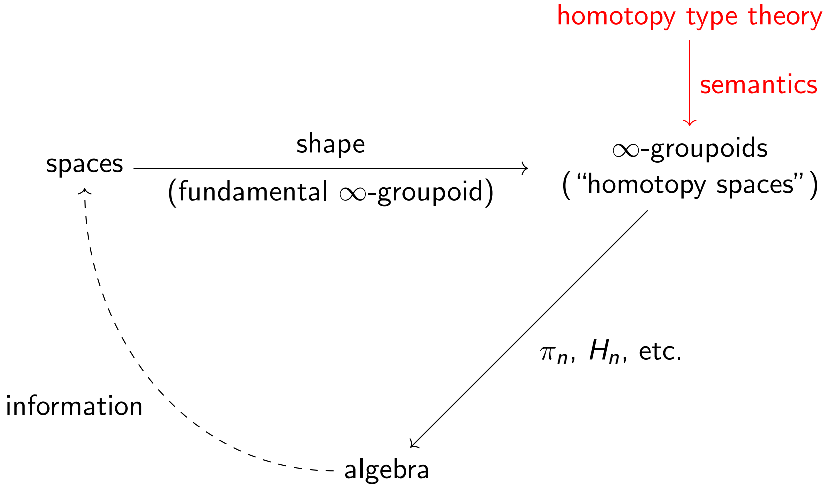
Classical algebraic topology



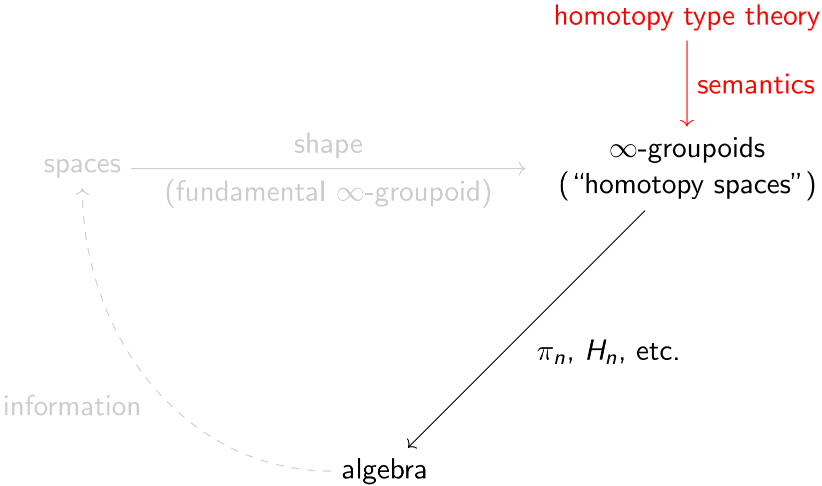
Modern algebraic topology



Homotopy type theory



Homotopy type theory

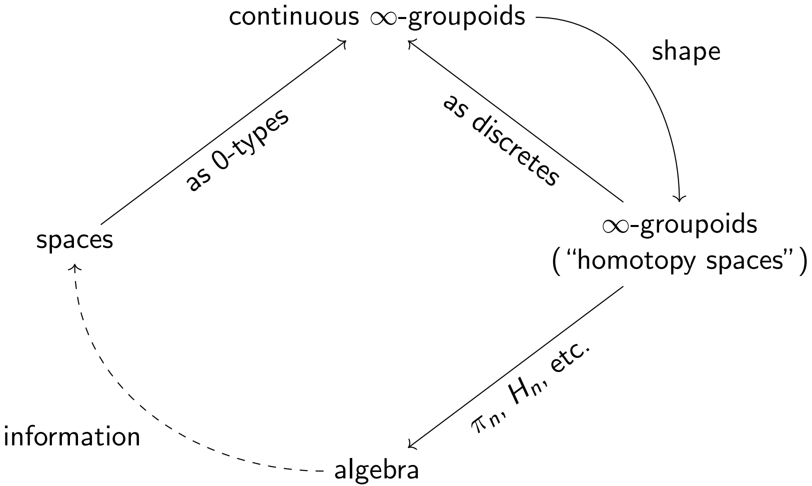


Geometry vs Homotopy theory

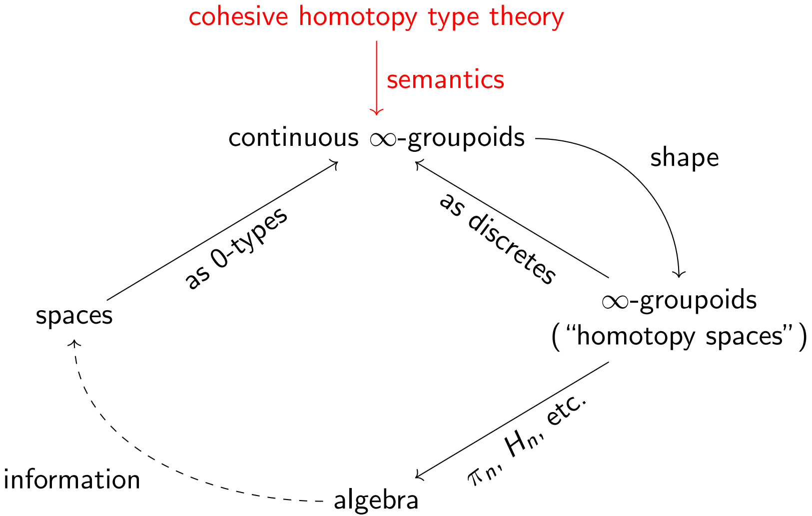
Many of the classical applications of algebraic topology require passing back and forth between treating a topological space up to homeomorphism and up to homotopy equivalence:

1. Brouwer's fixed-point theorem
2. The Borsuk-Ulam theorem
3. The fundamental theorem of algebra
4. The hairy ball theorem
5. The Lefschetz fixed-point theorem

Cohesive algebraic topology



Cohesive homotopy type theory



Continuous ∞-groupoids

Idea

A continuous ∞-groupoid is an ∞-groupoid with compatible topologies on the set of k-morphisms for all k.

Examples

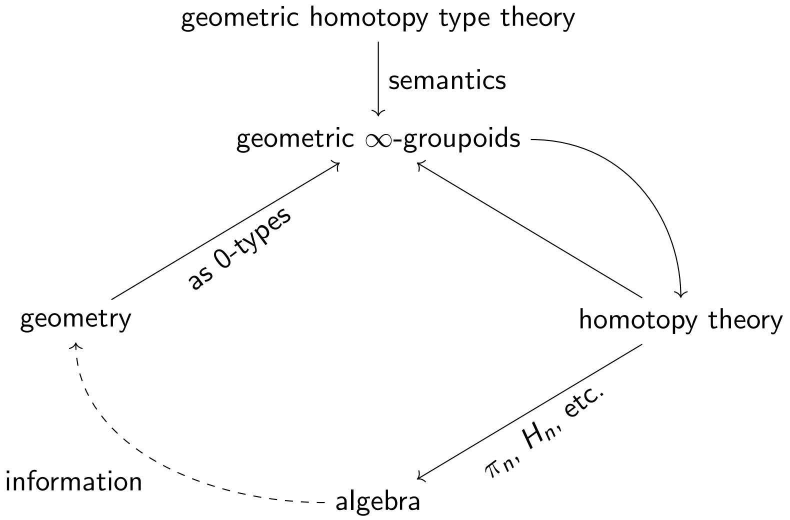
- An ordinary topological space of objects, with only identity k-morphisms for $k > 0$.
- An ordinary ∞-groupoid, with the discrete topology in all dimensions.
- An ordinary ∞-groupoid with the *indiscrete* topology.
- The delooping of a topological group G, with one object, with G as the *space* of 1-morphisms, and only k-identities for $k > 1$.

Other kinds of geometry

We also want to use homotopy theory to study more "geometric" contexts, like:

1. Smooth geometry
2. Super geometry
3. Equivariant geometry
4. Algebraic geometry
5. ...

Geometric homotopy type theory



Geometric ∞-groupoids

Idea

A *geometric ∞-groupoid* is an ∞-groupoid with compatible "geometries" on the set of k-morphisms for all k.

Definition

A geometric ∞-groupoid is an ∞-sheaf on some ∞-site of geometric spaces.

In other words:

1. Start with some small category $\mathscr{C}$ of geometric spaces.
2. "Freely" add ∞-colimits (pass to its ∞-presheaf category).
3. Force some "good" colimits that existed in $\mathscr{C}$ (e.g. unions of open covers) to coincide with the free ones.

The category of geometric ∞-groupoids is then an ∞-topos, and hence* interprets homotopy type theory.

Some geometries

Some geometric ∞-sites:

1. Cartesian spaces $\mathbb{R}^n$ and continuous maps
2. Cartesian spaces $\mathbb{R}^n$ and smooth maps
3. "Infinitesimally thickened" cartesian spaces
4. "Super" cartesian spaces
5. Affine schemes with Zariski, étale, Nisnevich, etc. covers

Other models of geometric HoTT include:

1. Global equivariance (actions of "all groups at once")
2. Parametrized spaces
3. Parametrized spectra
4. Excisive functors

Outline

What is Geometric HoTT?

Answer #1

We “expand the universe” of HoTT to include geometric ∞-groupoids in addition to ordinary ones.

Answer #2

We realize that the HoTT we’ve been doing all along *might as well* have been talking about geometric ∞-groupoids in addition to ordinary ones.

Adding homotopy to type theory

Ordinary type theory (for a mathematican)

- Intuition: *types as sets, terms as functions*.

Homotopy type theory

- New intuition: *types as ∞-groupoids, terms as functors*.
- Detect their ∞-groupoid structure with the identity type.
- The old intuition is still present in the 0-types.

Adding geometry to type theory

Ordinary type theory

- Intuition: *types as sets, terms as functions*.

Synthetic geometry

- New intuition: *types as geometric objects*.
- Detect their geometric structure in various ways.
- The old intuition is still present in the discrete objects.

Adding geometry AND homotopy to type theory

Geometric homotopy type theory

New intuition: *types as geometric ∞-groupoids*.

Every type has both ∞-groupoid structure and geometric structure. Either, both, or neither can be trivial.

Example

- The higher inductive S^1 has nontrivial higher structure ($\Omega S^1 = \mathbb{Z}$), but is geometrically discrete (no geometry).
- If $\mathbb{A}^1$ is a "line" (e.g. the real numbers), then $\mathbb{S}^1 = \left\{ (x, y) : \mathbb{A}^2 \;\middle|\; x^2 + y^2 = 1 \right\}$ has trivial higher structure (is a 0-type), but nontrivial geometry.

Usually, S^1 and $\mathbb{S}^1$ have the same "shape".

Outline

Towards geometric HoTT

So that's the idea of geometric HoTT: ordinary HoTT might as well always have been talking about geometric objects.

But to get any real information about geometry out of this, we need to use something about geometric ∞-groupoids that's not true in ordinary HoTT.

One approach is to assume axioms. Another, pioneered by Lawvere on the semantic side, is to equip type theory with modalities, i.e. systems of adjoint functors.

Discrete objects

There is always an adjunction of $(\infty, 1)$-categories:

$$\infty\text{-groupoids} \underset{p_\star}{\overset{p^\star}{\rightleftarrows}} \text{geometric } \infty\text{-groupoids}$$

- $p_\star(X)$ is the underlying ∞-groupoid of X (geometry forgotten)
- $p^\star(A)$ is A with "discrete geometry" (e.g. discrete topology).

Usually, $p^\star$ is fully faithful, so the discrete ∞-groupoids are a coreflective subcategory.

- We denote the reflector $p^\star p_\star$ by $\flat$.
- A map $\flat X \to Y$ is a map on underlying ∞-groupoids (no geometry).
- Every map out of a discrete ∞-groupoid is geometric.

Codiscrete objects

Definition

Y is codiscrete if every map into it is geometric.

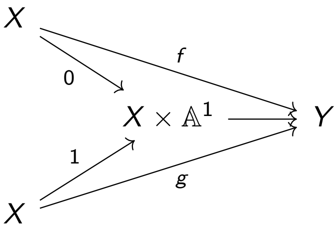
$$\frac{\flat X \to Y}{X \to \sharp Y}$$

Often, the codiscrete objects are reflective, by an adjunction $\flat \dashv \sharp$ that induces equivalences

discrete geometric ∞-groupoids	$\simeq$	ordinary ∞-groupoids	$\simeq$	codiscrete geometric ∞-groupoids

Homotopical objects

Let $\mathbb{A}^1$ be a "geometric line" (e.g. $\mathbb{R}$), with points $0, 1 : \mathbb{A}^1$.
A geometric homotopy $f \overset{\text{geo}}{\sim} g$ is a map $X \times \mathbb{A}^1 \to Y$ such that



Definition

An object Y is homotopical (or *geometric homotopy local*) if $Y \to (\mathbb{A}^1 \to Y)$ is an equivalence.

In particular, then $f \overset{\text{geo}}{\sim} g$ implies $f = g$: geometric homotopies imply synthetic homotopies.

Shape

By nullification at $\mathbb{A}^1$ there is a shape modality $\int$ whose modal types are the homotopical ones.

($\int$ is an "esh", the IPA voiceless postalveolar fricative — English *sh*)

- If geometry is "locally contractible", homotopical = discrete. This is cohesive HoTT: discretes are both reflective and coreflective, and we have an adjoint triple $\int \dashv \flat \dashv \sharp$.
  - Continuous ∞-groupoids
  - Smooth, differential, super, etc. ∞-groupoids
  - Global equivariance, parametrized spectra, excisive functors, etc.
- But sometimes this isn't true.
  - Algebraic geometry: homotopical objects are *motivic spaces*.

Fundamental ∞-groupoids

Idea: $\int$ turns geometric paths into homotopical ones.

Example

Suppose the geometric circle $\mathbb{S}^1$ can be written as "$\mathbb{A}^1/\mathbb{Z}$", i.e. a coequalizer

$$\mathbb{A}^1 \rightrightarrows \mathbb{A}^1 \longrightarrow \mathbb{S}^1.$$

Then $\int \mathbb{S}^1 = \int S^1$, since both are a coequalizer of $1 \rightrightarrows 1$ in the category of homotopical objects.

In cohesive HoTT, with $\int \dashv \flat \dashv \sharp$:

- The homotopical (= discrete) objects are coreflective, hence closed under colimits. Thus $\int \mathbb{S}^1 = \int S^1 = S^1$.
- More generally, $\int$ computes the fundamental ∞-groupoid.

(Algebraically, $\int$ is the "motivic fundamental ∞-groupoid".)

Outline

Semantic reflective subuniverses

Because a reflective subuniverse is a map $\lozenge : \mathcal{U} \to \mathcal{U}$, it induces a reflective subcategory not just of the category of types but of all of its slice categories (a reflective subfibration):

$$
\begin{array}{ccc}
Y & \longrightarrow & \widetilde{U} \\
\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\
X & \longrightarrow & U
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{ccccc}
\lozenge_X Y & & \longrightarrow & & \widetilde{U} \\
\downarrow & \lrcorner & & & \downarrow{\scriptstyle \pi} \\
X & \longrightarrow & U & \xrightarrow[\lozenge]{} & U
\end{array}
$$

The internal construction of localization means that all accessible reflective subcategories can be extended to some reflective subfibration.

The monadic modalities (Σ-closed reflective subuniverses) correspond to stable factorization systems.

The problem of discrete coreflection

- $\int$ is a monadic modality, definable purely inside type theory as a nullification at $\mathbb{A}^1$.
- ($\sharp$ too, although its generators are less obvious.)
- $\flat$ is a comonadic modality, but cannot be defined internally.

Theorem

The only internal "coreflective subuniverses" are $\Box(X) = X \times P$ for some proposition P.

Proof.

Given $\Box$, let $P = \Box 1$.

- The map $X \to 1$ yields $\Box X \to \Box 1$, hence (with the counit) $\Box X \to X \times \Box 1$.
- For any $x : X$ we have $x : 1 \to X$, hence $\Box x : \Box 1 \to \Box X$. This defines $X \times \Box 1 \to \Box X$.

Then show these are inverses. $\Box$

The solution to discrete coreflection

First Solution

$\flat$ can only be applied in the empty context.

Semantically: discrete objects are a coreflective subcategory of geometric ∞-groupoids, but not of all its slice categories.

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Better Solution

$\flat$ can only be applied when everything in the context is discrete.

Semantically: discrete objects are a coreflective subcategory of geometric ∞-groupoids, considered as *indexed* over ordinary ∞-groupoids.

Why modal type theory? (for type theorists)

$$\frac{x : \flat A \vdash C}{x : \flat A \vdash \flat C} \qquad \text{or} \qquad \frac{x :: A \mid \cdot \vdash C}{x :: A \mid y : B \vdash \flat C}$$

- Literally requiring types in the context to be of the form $\flat A$ breaks the admissibility of substitution.
- Instead we "judgmental-ize" it with a formalism of "crisp variables" $x :: A$.
- The modality $\flat$ internalizes the judgmental $::$ in the same way that the cartesian product "internalizes" the judgmental comma

$$\frac{A, B \vdash C}{A \times B \vdash C.}$$

Why modal type theory? (for category theorists)

$\mathscr{E}$ = a model category for geometric ∞-groupoids
$\mathscr{S}$ = a model category for ordinary ∞-groupoids

$$\mathscr{S} \underset{p_\star}{\overset{p^\star}{\rightleftarrows}} \mathscr{E} \qquad\qquad p^\star \dashv p_\star$$

- Regard $\mathscr{E}$ as indexed over $\mathscr{S}$ via $p^\star$: for $A \in \mathscr{S}$, the A-indexed objects are the slice over $p^\star A$.
- All ordinary types (Σ, Π, etc.) denote structure of $\mathscr{E}$.
- $x : A$ in the context means working in the slice $\mathscr{E}/A$.
- $x :: A$ in the context means working indexed over $p_\star A$, hence in the slice $\mathscr{E}/p^\star p_\star A$.
- Types are fibrant objects, but $p^\star p_\star A$ may not be fibrant. $\flat A$ is its fibrant replacement.

Crisp variables

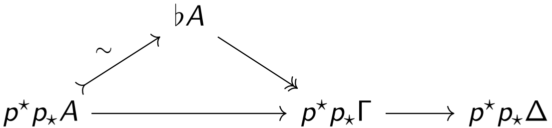
- We refer to $x :: A$ as a crisp variable.
  If $x :: P$ with P a proposition, we say P holds crisply.
- For emphasis, we sometimes call $x : A$ a cohesive variable.
- $x :: A$ is a *stronger hypothesis* than $x : A$.
  Something which holds crisply also holds cohesively.
- A crisp term or crisp conclusion is one that only uses crisp variables/hypotheses.
- We can substitute:
  - crisp terms for crisp variables
  - crisp terms for cohesive variables
  - cohesive terms for cohesive variables

  but not cohesive terms for crisp variables.

$\flat$-formation and introduction

A general judgment has the form $\Delta \mid \Gamma \vdash \mathcal{J}$, where Δ is a context of crisp variables and Γ a context of cohesive ones. Semantically this is a fibration $\Gamma \twoheadrightarrow p^\star p_\star \Delta$.

- A type $\Delta \mid \Gamma \vdash A : \mathsf{Type}$ is a fibration $A \twoheadrightarrow \Gamma \twoheadrightarrow p^\star p_\star \Delta$.
- Applying $p^\star p_\star$ (note $p_\star p^\star = \mathrm{Id}$) and fibrantly replacing, we get



- In syntax this becomes

$$\frac{\Delta \mid \Gamma \vdash A : \mathsf{Type}}{\Delta, \Gamma \mid \cdot \vdash \flat A : \mathsf{Type} \qquad \Delta, \Gamma, x :: A \mid \cdot \vdash x^\flat : \flat A}$$

$\flat$-formation and introduction, type-theoretically

$$\frac{\Delta \mid \Gamma \vdash A : \mathsf{Type}}{\Delta, \Gamma \mid \cdot \vdash \flat A : \mathsf{Type} \qquad \Delta, \Gamma, x :: A \mid \cdot \vdash x^\flat : \flat A}$$

1. No point to separating Γ from Δ, since $\Delta \mid \Gamma \vdash A : \mathsf{Type}$ is a stronger hypothesis than $\Delta, \Gamma \mid \cdot \vdash A : \mathsf{Type}$.

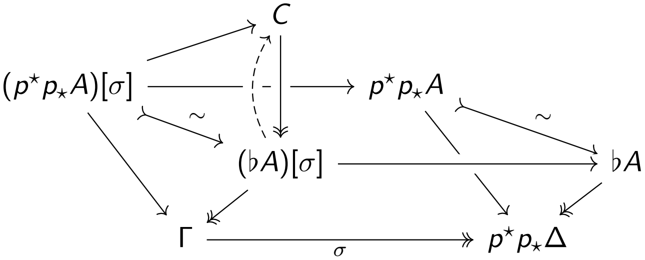
$$\frac{\Delta \mid \cdot \vdash A : \mathsf{Type}}{\Delta \mid \cdot \vdash \flat A : \mathsf{Type} \qquad \Delta, x :: A \mid \cdot \vdash x^\flat : \flat A}$$

2. Allow "fully general contexts" in the conclusion, including substituting a crisp term for the crisp variable x:

$$\frac{\Delta \mid \cdot \vdash A : \mathsf{Type}}{\Delta \mid \Gamma \vdash \flat A : \mathsf{Type}} \qquad \frac{\Delta \mid \cdot \vdash A : \mathsf{Type} \qquad \Delta \mid \cdot \vdash a : A}{\Delta \mid \Gamma \vdash a^\flat : \flat A}$$

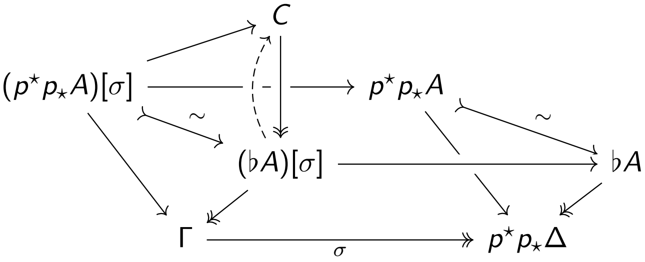
Summary: From "inside" $\flat(-)$ or $(-)^\flat$, no cohesive variables from "outside" are visible.

$\flat$-elimination and computation



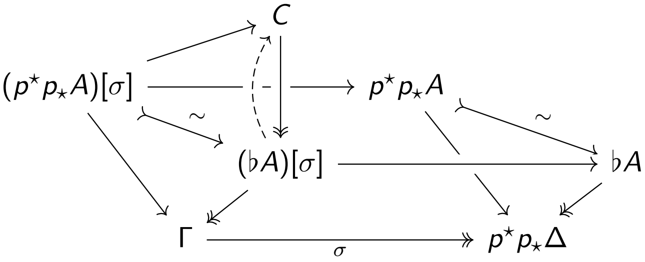
$$\frac{\Delta \mid \Gamma, x : \flat A \vdash C : \mathsf{Type} \qquad \Delta, u :: A \mid \Gamma \vdash c : C[u^\flat / x]}{\Delta \mid \Gamma, x : \flat A \vdash (\mathsf{let}\ u^\flat \coloneqq x\ \mathsf{in}\ c) : C}$$

♭-elimination and computation



$$\frac{\Delta \mid \Gamma, x : \flat A \vdash C : \mathsf{Type} \qquad \Delta, u :: A \mid \Gamma \vdash c : C[u^\flat / x] \qquad \Delta \mid \Gamma \vdash b : \flat A}{\Delta \mid \Gamma \vdash (\mathsf{let}\ u^\flat := b\ \mathsf{in}\ c) : C[b/x]}$$

$\flat$-elimination and computation



$$\frac{\Delta \mid \Gamma, x : \flat A \vdash C : \mathsf{Type} \qquad \Delta, u :: A \mid \Gamma \vdash c : C[u^\flat / x] \qquad \Delta \mid \Gamma \vdash b : \flat A}{\Delta \mid \Gamma \vdash (\mathsf{let}\ u^\flat := b\ \mathsf{in}\ c) : C[b/x]}$$

$$(\mathsf{let}\ u^\flat := a^\flat\ \mathsf{in}\ c) \equiv c[a/u]$$

The count of $\flat$

Remember that $\flat$ is a coreflection, so it should have a counit.

$$\begin{array}{ccc} p^\star p_\star A & \longrightarrow & A \\ {\scriptstyle\sim}\big\downarrow & \nearrow & \big\downarrow \\ \flat A & \twoheadrightarrow & p^\star p_\star \Delta \end{array}$$

$(-)_\flat : \flat A \to A$ is defined by $x_\flat \equiv (\text{let } u^\flat := x \text{ in } u)$.

Note u is a crisp variable, but gets used as a cohesive one; this corresponds to the strict counit $p^\star p_\star A \to A$.

For $v :: A$ we have ${v^\flat}_\flat \equiv (\text{let } u^\flat := v^\flat \text{ in } u) \equiv v$. Note this only makes sense for crisp v, since only then can we even write $v^\flat$.

Properties of $\flat$

Lemma (Uniqueness principle)

For $f : \prod_{x:\flat A} C(x)$ *and* $x : \flat A$ *we have* $(\mathsf{let}\ u := x\ \mathsf{in}\ f(u^\flat)) = f(x)$.

Proof.

By $\flat$-elimination, we may assume x is $v^\flat$ for some $v :: A$.
But then $(\mathsf{let}\ u := v^\flat\ \mathsf{in}\ f(u^\flat)) \equiv f(v^\flat)$ by computation. □

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Lemma (Commuting $\flat$ with itself)

$(\mathsf{let}\ v^\flat := (\mathsf{let}\ u^\flat := M\ \mathsf{in}\ N)\ \mathsf{in}\ P) = (\mathsf{let}\ u^\flat := M\ \mathsf{in}\ (\mathsf{let}\ v^\flat := N\ \mathsf{in}\ P))$.

Proof.

By $\flat$-elimination we may assume M is $w^\flat$ for some $w :: A$.
But then both sides reduce to $(\mathsf{let}\ v^\flat := N[w/u]\ \mathsf{in}\ P)$. □

$\flat$ is a functor

For $f :: A \to B$, define $\flat f : \flat A \to \flat B$ by

$$\flat f(x) \equiv (\text{let } u^\flat := x \text{ in } f(u)^\flat)$$

Then for $g : B \to C$ we have

$$\begin{aligned}
\flat g(\flat f(x)) &\equiv \text{let } v^\flat := (\text{let } u^\flat := x \text{ in } f(u)^\flat) \text{ in } g(v)^\flat \\
&= \text{let } u^\flat := x \text{ in } (\text{let } v^\flat := f(u)^\flat \text{ in } g(v)^\flat) \\
&\equiv \text{let } u^\flat := x \text{ in } g(f(u))^\flat \\
&\equiv \flat(g \circ f)(x).
\end{aligned}$$