Semantics of higher modalities

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Geometry in Modal Homotopy Type Theory workshop
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Outline

1. Higher topos semantics
2. Spatial type theory
3. Cohesive type theory
4. Examples
5. Other modal type theories

Geometric HoTT: Review

Definition

A geometric ∞-groupoid is an ∞-sheaf on some $(\infty, 1)$-site of "geometric spaces".

Definition

An $(\infty, 1)$-site is a small $(\infty, 1)$-category $\mathfrak{C}$ equipped with an accessible left exact localization of $[\mathfrak{C}^{\mathrm{op}}, \infty\mathfrak{Gpd}]$.
The local types are the $(\infty, 1)$-sheaves.
The category of $(\infty, 1)$-sheaves is a Grothendieck $(\infty, 1)$-topos.

We want to interpret type theory in the $(\infty, 1)$-topos of geometric ∞-groupoids, or more generally in any $(\infty, 1)$-topos.

Modeling type theory in higher toposes

$$
\begin{array}{ccc}
\text{syntactic type theory} & \xrightarrow{(1)} & \begin{array}{c}\text{contextual categories/}\\ \text{categories with families/}\\ \text{natural models/}\ldots\end{array} \\
 & & \big\downarrow (2) \\
 & & \text{Quillen model categories} \\
 & & \big\downarrow (3) \\
 & & (\infty, 1)\text{-categories}
\end{array}
$$

1. is the "initiality principle": lots of bookkeeping that should be written out more systematically, but is undoubtedly true.
2. is a coherence theorem: Voevodsky "global universes", Lumsdaine–Warren "local universes", etc.
3. $\cdots$

Modeling homotopy type theory in simplicial presheaves

Present an $(\infty,1)$-topos as a left exact left Bousfield localization $L_S[\mathscr{C}^{\mathrm{op}}, \mathscr{S}]$ of the injective model structure on simplicial presheaves over some small simplicial category $\mathscr{C}$.

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- Σ*-types,* Π*-types, and identity types.*
- *Strict univalent universes, closed under the above.*
- *Pushouts, localizations,* W*-types, and other HITs.*

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Corollary

We can interpret most of homotopy type theory in any Grothendieck $(\infty, 1)$-topos.

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(Closure of the universes under HITs is work in progress.)

Sketch of proof I: type formers

1. Cofibrations in $L_S[\mathscr{C}^{\mathrm{op}}, \mathscr{S}]$ are the monos, hence pullback-stable.
2. $L_S[\mathscr{C}^{\mathrm{op}}, \mathscr{S}]$ is right proper: w.e.s pullback-stable along fibrations.
3. $\therefore$ acyclic cofibrations are pullback-stable along fibrations. Can interpret identity types by path objects (Awodey–Warren).
4. $L_S[\mathscr{C}^{\mathrm{op}}, \mathscr{S}]$ is locally cartesian closed.
5. $\therefore$ fibrations and acyclic fibrations are closed under dependent product along fibrations. Thus we can interpret Π-types.
6. Interpret Σ-types by composing fibrations.
7. Interpret HITs by mixing fibrant replacement with free algebras constructions (Lumsdaine–S.).

This has all been known for some years.

Sketch of proof II: injective fibrations

The injective fibrations are, by definition, the maps having the right lifting property with respect to all pointwise acyclic cofibrations. But this is unhelpful for constructing a universe in general.

Lemma

A pointwise fibration $f : X \twoheadrightarrow Y$ *in* $[\mathscr{C}^{\mathrm{op}}, \mathscr{S}]$ *has a relative pseudomorphism classifier* $\mathcal{R}f \to Y$ *and a natural bijection between*

1. *(Strict) natural transformations* $A \to \mathcal{R}f$.
2. *Homotopy coherent transformations* $A \rightsquigarrow X$ *such that the composite* $A \rightsquigarrow X \to Y$ *is strict.*

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2. *Homotopy coherent transformations $A \rightsquigarrow X$ such that the composite $A \rightsquigarrow X \to Y$ is strict.*

Lemma

$f : X \to Y$ in $[\mathscr{C}^{\mathrm{op}}, \mathscr{S}]$ is an injective fibration if and only if

1. *it is a pointwise fibration, and*
2. *the canonical map $X \to \mathcal{R}f$ has a retraction over Y.*

Sketch of proof III: Presheaf universes

Define a *semi-algebraic injective fibration* to be a pointwise fibration equipped with a retraction of $X \to \mathcal{R}f$.

Lemma

In $[\mathscr{C}^{\mathrm{op}}, \mathscr{S}]$, a universe can be "defined" by

$$U(c) = \Big\{ \textit{small semi-algebraic injective fibrations over } \mathscr{C}(-, c) \Big\}.$$

- Choose an inaccessible cardinal to define "small"
- Need to choose iso representatives, etc., to strictify
- Semi-algebraicity ensures fibrations can be glued together to make a universal one over U.

Sketch of proof IV: Sheaf universes

Lemma

Any accessible left exact localization of $[\mathscr{C}^{\mathrm{op}}, \mathscr{S}]$ *induces an accessible lex modality in its internal type theory.*

Proof.

By Anel–Biedermann–Finster–Joyal (2019, forthcoming), we can present it by localization at a family that remains lex on pullback to all slice categories. $\square$

Lemma (Rijke–S.–Spitters)

If $\lozenge$ *is an accessible lex modality in type theory, then* $\mathcal{U}_\lozenge := \sum_{X:\mathcal{U}} \mathsf{IsModal}_\lozenge(X)$ *is* $\lozenge$*-modal.*

Modeling type theory in higher toposes

Corollary

We can interpret most of homotopy type theory in any Grothendieck $(\infty, 1)$*-topos.*

Modeling $\flat$ in higher toposes

Let $\mathscr{C}$ be a 1-site with a terminal object, and $\mathscr{E} = L_S[\mathscr{C}^{\mathrm{op}}, \mathscr{S}]$. Then we have an adjoint quadruple:

$$\begin{tikzcd} \mathscr{E} \arrow[d, shift right=6, "p_!"'] \arrow[d, shift left=2, "p_\star"] \\ \mathscr{S} \arrow[u, shift right=2, "p^\star"] \arrow[u, shift left=6, "p^\#"'] \end{tikzcd}$$

$p_!(A) = \operatorname{colim} X$
$p^\star(A) =$ the constant presheaf at A
$p_\star(X) = \lim X = X(1)$
$p^\#(A)(c) = A^{\mathscr{C}(1,c)}$

$p^\star$ and $p^\#$ are fully faithful, & have a map $p^\star \to p^\#$.
$p^\star$ preserves cofibrations and w.e.s, so $p^\star \dashv p_\star$ is Quillen.
$p^\star p_\star$ doesn't preserve fibrations: fibrantly replace it to get $\flat$.

A disclaimer

Unlike for ordinary type theory, the precise categorical semantics of modal dependent type theories, and its relation to Quillen model categories (including coherence theorems), is still work in progress by various people.

What I'm presenting here is an informal sketch of aspects of Quillen model category theory that I think are likely to figure importantly in that semantics when it is fully developed, and that can help to motivate the rules of modal type theory for a homotopy theorist.

Modeling $\sharp$ in higher toposes

$$\begin{tikzcd} \mathscr{E} \arrow[d, shift right=6, "p_!"'] \arrow[d, shift right=2, phantom, "\vdash"] \arrow[d, shift left=2, "p_\star"] \arrow[d, shift left=4, phantom, "\dashv"] \\ \mathscr{S} \arrow[u, shift right=2, "p^\star"'] \arrow[u, shift left=6, "p^\#"'] \end{tikzcd}$$

$p_!(A) = \operatorname{colim} X$
$p^\star(A) =$ the constant presheaf at A
$p_\star(X) = \lim X = X(1)$
$p^\#(A)(c) = A^{\mathscr{C}(1,c)}$

$p_\star$ always preserves cofibrations. If $1 \in \mathscr{C}$ has no nontrivial covers, then $p_\star$ preserves weak equivalences, so $p_\star \dashv p^\#$ is Quillen.

Thus we get an adjoint triple $\mathbb{L}p^\star \dashv p_\star \dashv \mathbb{R}p^\#$ on $(\infty, 1)$-categories: a local $(\infty, 1)$-topos.

Now $p^\# p_\star$ preserves fibrations; no need for fibrant replacement.

Outline

Spatial type theory

We introduce $\sharp$ as the "negative dual" of $\flat$, analogously to the adjunction between $\times$ and $\rightarrow$.

$$\frac{A, B \vdash C}{A \times B \vdash C} \qquad \frac{A, B \vdash C}{A \vdash B \rightarrow C}$$

$$\frac{A \mid \cdot \vdash B}{\cdot \mid \flat A \vdash C} \qquad \frac{A \mid \cdot \vdash B}{\cdot \mid A \vdash \sharp C}$$

$\sharp$-formation and introduction

$$\begin{tikzcd}
A \arrow[r, "(-)^\sharp"] \arrow[d, two heads] & \sharp A \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & p^{\#} p_\star A \arrow[d, two heads] \\
p^\star p_\star \Gamma \arrow[r] \arrow[dr] & \Gamma \arrow[r] \arrow[d, two heads] & p^{\#} p_\star \Gamma = p^{\#} p_\star p^\star p_\star \Gamma \arrow[d, two heads] \\
& p^\star p_\star \Delta \arrow[r] & p^{\#} p_\star \Delta = p^{\#} p_\star p^\star p_\star \Delta
\end{tikzcd}$$

$$\frac{\Delta, \Gamma \mid \cdot \vdash A : \mathsf{Type}}{\Delta \mid \Gamma \vdash \sharp A : \mathsf{Type}} \qquad \frac{\Delta, \Gamma \mid \cdot \vdash a : A}{\Delta \mid \Gamma \vdash a^\sharp : \sharp A}$$

Summary: From “inside” $\sharp(-)$ or $(-)^\sharp$, the “outside” cohesive variables can be treated as crisp.
(Recall: Inside $\flat(-)$ or $(-)^\flat$, outside cohesive variables are invisible.)

$\sharp$-elimination, computation, and uniqueness

$$
\begin{tikzcd}
A \arrow[r, "(-)^\sharp"] \arrow[d, two heads] & \sharp A \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & p^\# p_\star A \arrow[d, two heads] \\
p^\star p_\star \Gamma \arrow[r] \arrow[dr] & \Gamma \arrow[r] \arrow[d, two heads] & p^\# p_\star \Gamma = p^\# p_\star p^\star p_\star \Gamma \arrow[d, two heads] \\
 & p^\star p_\star \Delta \arrow[r] & p^\# p_\star \Delta = p^\# p_\star p^\star p_\star \Delta
\end{tikzcd}
$$

$p_\star$ inverts all the horizontal arrows in this diagram.
Thus $p^\star p_\star \sharp A \cong p^\star p_\star A \to A$, hence $x :: \sharp A \vdash x_\sharp : A$, or

$$\frac{\Delta \mid \cdot \vdash a : \sharp A}{\Delta \mid \Gamma \vdash a_\sharp : A} \qquad (a_\sharp)^\sharp \equiv a \qquad (a^\sharp)_\sharp \equiv a$$

No fibrant replacement is involved, so both computation and uniqueness rules can be judgmental.

Comparing $\flat$ and $\sharp$

Lemma

For any A :: Type, there are natural equivalences

$$\sharp\flat A \simeq \sharp A \qquad \textit{and} \qquad \flat A \simeq \flat\sharp A.$$

Proof.

$$(\lambda y.{y_{\sharp\flat}}^{\sharp}) : \sharp\flat A \to \sharp A \qquad (\lambda x.{x_{\sharp}}^{\flat\sharp}) : \sharp A \to \sharp\flat A$$
$$(\lambda x.\mathsf{let}\ u^{\flat} := x\ \mathsf{in}\ u^{\sharp\flat}) : \flat A \to \flat\sharp A$$
$$(\lambda y.\mathsf{let}\ v^{\flat} := y\ \mathsf{in}\ {v_{\sharp}}^{\flat}) : \flat\sharp A \to \flat A$$

And then some calculation. □

Hence $\sharp\flat A \simeq A$ when A is codiscrete ($A \simeq \sharp A$),
and $\flat\sharp B \simeq B$ when B is discrete ($B \simeq \flat B$).

Other facts about spatial type theory

- $\sharp$ is a left exact monadic modality
- $(x^\sharp = y^\sharp) \simeq \sharp(x = y)$ for $x, y : A$.
- $\flat$ is a left exact comonadic modality
- $(x^\flat = y^\flat) \simeq \flat(x = y)$ for $x, y :: A$.
- $\flat(\flat A \to B) \simeq \flat(A \to \sharp B)$, i.e. $\flat \dashv \sharp$ "crisply"

Outline

Presenting cohesive $(\infty, 1)$-toposes

$$\begin{tikzcd}
\mathscr{E} \arrow[d, shift right=9, "p_!"'] \arrow[d, phantom, shift right=6, "\dashv"] \arrow[d, shift left=3, "p_\star"] \arrow[d, phantom, shift left=6, "\dashv"] \\
\mathscr{S} \arrow[u, shift right=3, "p^\star"'] \arrow[u, shift right=9, "p^\#"']
\end{tikzcd}$$

$p^\star(A) =$ the constant presheaf at A.

There's basically no chance that $p_! \dashv p^\star$ will ever be Quillen.
$p^\star A$ is almost never injectively fibrant, let alone local.

We could fibrantly replace $p^\star p_! X$ to get $\int X$, but that would clobber the context. A better solution is to construct it internally.

$(\infty, 1)$-cohesive sites

Definition (Schreiber)

A 1-site is ∞-cohesive if it has an irreducible terminal object, finite products, and every covering sieve has a contractible nerve.

Lemma

If $\mathscr{C}$ is ∞-cohesive, then TFAE for fibrant $X \in \mathscr{E}$:

1. *X is discrete, i.e. $X \simeq p^\star p_\star X$.*
2. *$X(U) \to X(V)$ is an equivalence in $\mathscr{S}$ for any $V \to U$ in $\mathscr{C}$.*
3. *$X(1) \to X(V)$ is an equivalence in $\mathscr{S}$ for any $V \in \mathscr{C}$.*
4. *$X(U) \to X(U \times V)$ is an equivalence in $\mathscr{S}$ for all $U, V \in \mathscr{C}$.*
5. *$X \to (\mathscr{C}(-, V) \to X)$ is an equivalence in $\mathscr{E}$ for all V in some set of objects that generate $\mathscr{C}$ under finite products.*

Often $\mathscr{C}$ is generated under finite products by one object $\mathbb{A}^1$ or $\mathbb{R}$.

Cohesive type theory

Axiom C0

There is a type family R_i such that a crisp type $X :: \text{Type}$ is discrete (i.e. $X \simeq \flat X$) if and only if it is R-null (i.e. each $X \to (R_i \to X)$ is an equivalence).

We can then construct $\int$ by nullifying at R, giving $\int \dashv \flat \dashv \sharp$.

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Spatial type theory with this axiom has conjectural semantics in local and stably locally ∞-connected $(\infty, 1)$-toposes,
a.k.a. cohesive $(\infty, 1)$-toposes:

- $(\infty, 1)$-toposes whose global sections geometric morphism $\mathscr{E} \to \mathscr{S}$ extends to an adjoint string $p_! \dashv p^\star \dashv p_\star \dashv p^\#$ where $p_!$ preserves finite products. (Lawvere, Johnstone, Schreiber)
- Categories of "geometric ∞-groupoids" locally modeled on a good class of "geometrically contractible" spaces.

Real-cohesive homotopy type theory

Continuous ∞-groupoids = sheaves on the site $\mathscr{C} = \{\mathbb{R}^n\}$ of cartesian spaces, continuous maps, and open covers.

- Models cohesive type theory
- Generated under products by $\mathbb{R} \in \mathscr{C}$
- The representable $\mathscr{C}(-, \mathbb{R})$ is also the Dedekind real numbers object in $\mathscr{E}$.

Axiom $R\flat$

A crisp type $X :: \text{Type}$ is discrete (i.e. $X \simeq \flat X$) if and only if it is $\mathbb{R}$-null (i.e. $X \to (\mathbb{R} \to X)$ is an equivalence), where $\mathbb{R}$ is the Dedekind real numbers.

Other kinds of cohesion

Axiom C1

Axiom C0 + for all i we have an element $r_i : R_i$.

This implies (and should be semantically equivalent to) punctual local contractibility: $\flat A \to \int A$ is surjective.

Axiom C1

Axiom C1 + some R_{i_0} is a set and has $r_0, r_1 : R_{i_0}$ with $r_0 \neq r_1$.

This implies (and should be semantically equivalent to) contractible codiscreteness: $\int \sharp A \cong \|A\|$ for crisp A.

Outline

Real-cohesive homotopy type theory

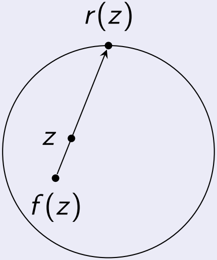
$$\begin{tikzcd}
\text{constructive analysis} \arrow[rr, "\int"] & & \text{synthetic homotopy theory} \arrow[ll, bend left, dashed, "\text{information}"]
\end{tikzcd}$$

Brouwer's fixed-point theorem (classical version)

Theorem

Any continuous map $f : \mathbb{D}^2 \to \mathbb{D}^2$ has a fixed point.

Proof.



Suppose $f : \mathbb{D}^2 \to \mathbb{D}^2$ has no fixed point. For any $z \in \mathbb{D}^2$, draw the ray from $f(z)$ through z to hit $\partial\mathbb{D}^2 = \mathbb{S}^1$ at $r(z)$. Then r is continuous, and retracts $\mathbb{D}^2$ onto $\mathbb{S}^1$. Hence $\pi_1(\mathbb{S}^1) = \mathbb{Z}$ is a retract of $\pi_1(\mathbb{D}^2) = 0$, a contradiction.

$\square$

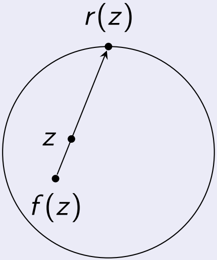
The real-cohesive version, first try

We work in real-cohesive HoTT, with $\int\mathbb{R} = 1$, hence $\int\mathbb{S}^1 = S^1$.

Theorem

Any function $f : \mathbb{D}^2 \to \mathbb{D}^2$ has a fixed point.

Attempted proof.



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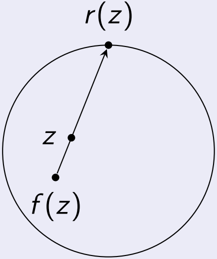

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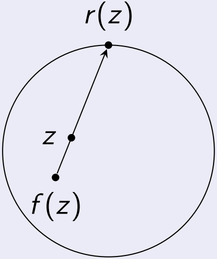

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□

Problems

There are two problems with this:

1. It's a proof by contradiction of a positive statement: the sort that's disallowed in constructive mathematics. But *cohesive* homotopy type theory is incompatible with excluded middle.
2. Even disregarding that, the assumption "f has no fixed point" tells us only that $f(z) \neq z$ for all z, whereas constructively, in order to draw the line connecting two points we need them to be *apart* (have a positive distance), not merely *unequal*.

Classicality axioms for cohesion

Flat excluded middle ($\flat$LEM)

For all $P :: \mathrm{Prop}$ we have $P + \neg P$.

"We can use proof by contradiction on crisp propositions."

Analytic Markov's Principle (AMP)

For $x, y : \mathbb{R}$, if $x \neq y$ then $|x - y| > 0$.

"Disequality implies apartness."

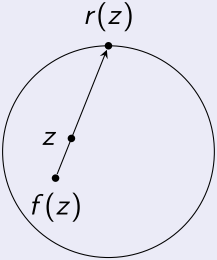
Both hold in the topos of continuous ∞-groupoids.

The real-cohesive version, second try

Theorem (Using $\flat$LEM and AMP)

Any function $f :: \mathbb{D}^2 \to \mathbb{D}^2$ has a fixed point.

Proof.



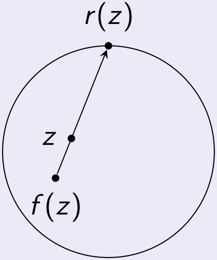
Given a crisp f, the statement is crisp, so we may use proof by contradiction. Suppose f has no fixed point. Then for any $z : \mathbb{D}^2$, we have $f(z) \neq z$, hence $d(z, f(z)) > 0$. So we can draw the ray from $f(z)$ through z to hit $\partial\mathbb{D}^2 = \mathbb{S}^1$ at $r(z)$. Then r retracts $\mathbb{D}^2$ onto $\mathbb{S}^1$. Hence $\int\mathbb{S}^1$ is a retract of $\int\mathbb{D}^2$. But $\mathbb{D}^2$ is a retract of $\mathbb{R}^2$, hence $\int\mathbb{D}^2$ is contractible, while $\int\mathbb{S}^1 = S^1$, which is not contractible. $\square$

The real-cohesive version, second try

Theorem (Using $\flat$LEM and AMP)

Any function $f :: \mathbb{D}^2 \to \mathbb{D}^2$ *has a fixed point.*

Proof.



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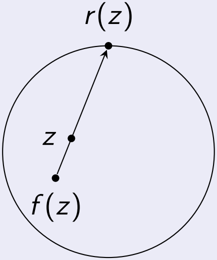
Proof.



Given a crisp f, the statement is crisp, so we may use proof by contradiction. Suppose f has no fixed point. Then for any $z : \mathbb{D}^2$, we have $f(z) \neq z$, hence $d(z, f(z)) > 0$. So we can draw the ray from $f(z)$ through z to hit $\partial\mathbb{D}^2 = \mathbb{S}^1$ at $r(z)$. Then r retracts $\mathbb{D}^2$ onto $\mathbb{S}^1$. Hence $\int\mathbb{S}^1$ is a retract of $\int\mathbb{D}^2$. But $\mathbb{D}^2$ is a retract of $\mathbb{R}^2$, hence $\int\mathbb{D}^2$ is contractible, while $\int\mathbb{S}^1 = S^1$, which is not contractible. $\square$

The crisp hypothesis $f :: \mathbb{D}^2 \to \mathbb{D}^2$ means the fixed point doesn't vary continuously with f.

Brouwer's other theorem

Theorem (Using AMP)

Any $f :: \mathbb{R} \to \mathbb{R}$ is ε-δ continuous at any $a :: \mathbb{R}$.

Idea of proof.

$$U = \{\, x \mid |f(x) - f(a)| < \varepsilon \,\} \qquad V = \{\, x \mid |f(x) - f(a)| > \tfrac{\varepsilon}{2} \,\}$$

- $\mathbb{R} = U \cup V$, so $\int U \sqcup^{\int(U \cap V)} \int V$ is contractible.
- Let U_0 be the "connected component" of $a \in U$, i.e. the set of points of U that are identified with a in $\int U$.
- Since $a \in \int U_0$ must be identified with "far away" points of $\int V$ in the pushout $\int U \sqcup^{\int(U \cap V)} \int V$, the set U_0 must contain a point b of V, which is thus apart from a.
- Everything between a and b must be in U, since U_0 is connected. Repeat on the other side of a. □

From connectedness to continuity

Corollary (Using $\flat$LEM and AMP)

$\flat(\mathbb{R} \to \mathbb{R})$ *is the subset of* ε-δ *continuous functions in* $\flat\mathbb{R} \to \flat\mathbb{R}$.

Real-cohesive homotopy type theory, plus $\flat$LEM and AMP, *determines the* ε-δ *notion of continuity*.

Outline

Cohesive type theories

Type Theory	Semantics
HoTT	∞-toposes
Spatial HoTT ($\flat \dashv \sharp$)	local ∞-toposes
Cohesive HoTT ($\int \dashv \flat \dashv \sharp$)	cohesive ∞-toposes
Real-cohesive HoTT ($\int \dashv \flat \dashv \sharp$, $\int$ generated by $\mathbb{R}$)	the ∞-topos of continuous ∞-groupoids
Differential cohesive HoTT ($\int \dashv \flat \dashv \sharp$ and $\Re \dashv \Im \dashv \&$)	differential cohesive ∞-toposes
Infinitesimal cohesive HoTT ($\int \dashv \flat \dashv \sharp$ with $\flat = \int = \sharp$)	infinitesimal cohesive ∞-toposes
etc.	etc.

Modal type theories in general

1. Specify a 2-category $\mathscr{M}$ of modes.
2. The "kinds of variables" are, roughly, the morphisms of $\mathscr{M}$.
3. Each morphism of $\mathscr{M}$ can induce an adjoint pair of modalities.

Example

For $\flat \dashv \sharp$, take $\mathscr{M}$ to have

- One object,
- One nonidentity morphism r,
- $rr = r$ and $\eta : 1 \Rightarrow r$, making r an idempotent monad.

- Licata–Shulman, "Adjoint logic with a 2-category of modes", 2016
- Licata–Shulman–Riley, "A Fibrational Framework for Substructural and Modal Logics", 2017
- Licata–Shulman–Riley, "A Fibrational Framework for Substructural and Modal Dependent Type Theories", 2019 (forthcoming)