Internal Languages for Higher Categories

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Outline

1. Internal languages for categories
2. Internal languages for higher categories
3. Research problems
The multiverse of mathematics

Theorem ("Gödel’s incompleteness theorem")

No strong, sensible formal system has a unique model.
The multiverse of mathematics

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Possible reactions:

1. Oh no! I thought I was studying one thing, but all this time I might have been studying something else entirely.
The multiverse of mathematics

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Possible reactions:

1. **Oh no!** I thought I was studying one thing, but all this time I might have been studying something else entirely.

2. **Hey cool!** All of my theorems are more general than I thought they were! That seems likely to be really useful.
Nonstandard models

Example

- **Formal system:** Grade-school arithmetic
- **Classical model:** The real numbers
- **Nonstandard models:** Other fields
Nonstandard models

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Example

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Example

- **Formal system**: Zermelo–Fraenkel set theory
- **Classical model**: The “real” universe of sets
- **Nonstandard models**: Alternative worlds of mathematics
Classical set-theoretic foundations are not ideal for working with alternative models. A better choice is type theory.

What you need to know about type theory for this talk

- Its basic objects are types, which are kind of like sets.
- Types contain terms, which are kind of like elements of sets.
- "a : A" means "the term a belongs to the type A".
- "(x : A) ⊢ (b : B)" means "assuming that x is a variable belonging to the type A, the term b belongs to the type B".
The “more general contexts” where we can interpret type theory are categories with appropriate structure.

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**Slogan**

Objects of any category “look internally” like sets.
Example

In mathematics built on type theory, a group is a type $G$ with terms

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\vdash (e : G) \\
(x : G), (y : G) \vdash (x \cdot y : G) \\
(x : G) \vdash (x^{-1} : G)
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Interpreted internally, we get

- In sets: a group.
- In topological spaces: a topological group.
- In manifolds: a Lie group.
- In sheaves: a sheaf of groups.
- In rings$^{\text{op}}$: a Hopf algebra.
Categorical semantics: really useful!

Theorem (proven in type theory)

The inversion map of a group is unique.
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*The inversion map of a group is unique.*

Therefore, the same is true of groups in any categorical model.

Example

The antipode of a Hopf algebra is unique.
Categorical semantics: really useful!

**Theorem** (proven in type theory)

*The inversion map of a group is unique.*

Therefore, the same is true of groups in any categorical model.

**Example**

The antipode of a Hopf algebra is unique.

**NB:** Not all classical reasoning is valid internally, e.g. the law of excluded middle and the axiom of choice usually fail. But there’s still a lot left.
So far, we’ve been talking about interpreting classically true statements in more general contexts.

We can also consider classically false statements (that is, statements which are false in the “originally intended” model) which are true in other interesting contexts.
Nonclassical axioms

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Examples

- All functions $\mathbb{R} \rightarrow \mathbb{R}$ are continuous.
- All functions $\mathbb{N} \rightarrow \mathbb{N}$ are computable.
- There exist real “infinitesimals” with which we can do calculus (synthetic differential geometry).
Outline

1. Internal languages for categories
2. Internal languages for higher categories
3. Research problems
A higher category is a category together with “higher morphisms” or “higher homotopies” between its morphisms. Thus it has:

- Objects $A, B, \ldots$
- Morphisms $A \to B,$
- 2-morphisms $\xymatrix{A & B}$
- 3-morphisms $\xymatrix{A & B}$
- \ldots

We will consider only $(\infty, 1)$-categories, where all the higher morphisms are “invertible”.
Higher categories

Examples

- Topological spaces
- Simplicial sets
- Chain complexes
- Categories
- Spectra (stable spaces)
- Simplicial sheaves

the “original model”, analogous to sets
The **absolutely magical fact** is that one of the most natural type theories is perfectly adapted to models in $(\infty, 1)$-categories.

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Homotopy type theory

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Slogan

Objects of an \((\infty, 1)\)-category “look internally” like \textit{spaces}, a.k.a. \textit{homotopy types}, a.k.a. \(\infty\)-\textit{groupoids}.
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We need to make the correspondence between type theory and higher categories precise (there are coherence issues).
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Complete models of type theory exist in any $(\infty, 1)$-topos.
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**Conjecture**
Complete models of type theory exist in any $(\infty, 1)$-topos.

What we have:

- An almost-complete model in any $(\infty, 1)$-topos (Lumsdaine–Warren and others).
- A complete model in the “standard” case of simplicial sets (Voevodsky).
- Complete models in a small class of presheaf $(\infty, 1)$-toposes (Shulman).
Problem 2

Question

Which parts of classical homotopy theory are still true internally?
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Some things we know:

- The law of excluded middle, the axiom of choice, all fail in general (just as for 1-categories).
- “Whitehead’s theorem” fails: a homotopy type is not determined by its homotopy groups (Lurie).
- But $\pi_1(S^1) = \mathbb{Z}$ (Shulman).
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Open Problem
Are all the homotopy groups of spheres determined by type theory? Or could they be different in different categories?
Question

What interesting nonclassical axioms can hold in higher categories?
Problem 3

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What interesting nonclassical axioms can hold in higher categories?

Example
A cohesive $(\infty, 1)$-topos is one whose objects are homotopy types equipped with some sort of topological or smooth structure. Its internal type-theory admits higher modalities, using which one can describe large chunks of differential cohomology and gauge theory (Schreiber–Shulman).
Thanks!