Internal Languages for Higher Categories

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Outline

The multiverse of mathematics

Theorem ("Gödel's incompleteness theorem")

No strong, sensible formal system has a unique model.

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1. Oh no! I thought I was studying one thing, but all this time I might have been studying something else entirely.
2. Hey cool! All of my theorems are more general than I thought they were! That seems likely to be really useful.

Nonstandard models

Example

- Formal system: Grade-school arithmetic
- Classical model: The real numbers
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Example

- Formal system: Zermelo–Fraenkel set theory
- Classical model: The "real" universe of sets
- Nonstandard models: Alternative worlds of mathematics

Type theory

Classical set-theoretic foundations are not ideal for working with alternative models. A better choice is type theory.

What you need to know about type theory for this talk

- Its basic objects are types, which are kind of like sets.
- Types contain terms, which are kind of like elements of sets.
- "$a : A$" means "the term a belongs to the type A".
- "$(x : A) \vdash (b : B)$" means "assuming that x is a variable belonging to the type A, the term b belongs to the type B".

Categorical semantics

The "more general contexts" where we can interpret type theory are categories with appropriate structure.

Type theory	Model in a category $\mathbf{C}$
Type $(A : \text{Type})$	Object $[\![A]\!] \in \mathbf{C}$
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Slogan

Objects of any category "look internally" like sets.

Categorical semantics: example

Example

In mathematics built on type theory, a group is a type G with terms

$$\begin{aligned} &\vdash\ (e : G) \\ (x : G), (y : G) &\vdash\ (x \cdot y : G) \\ (x : G) &\vdash\ (x^{-1} : G) \end{aligned}$$

satisfying the appropriate axioms.

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Interpreted internally, we get

- In sets: a group.
- In topological spaces: a topological group.
- In manifolds: a Lie group.
- In sheaves: a sheaf of groups.
- In rings$^{\mathrm{op}}$: a Hopf algebra.

Categorical semantics: really useful!

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NB: Not all classical reasoning is valid internally, e.g. the law of excluded middle and the axiom of choice usually fail. But there's still a lot left.

Nonclassical axioms

- So far, we've been talking about interpreting classically true statements in more general contexts.
- We can also consider classically false statements (that is, statements which are false in the "originally intended" model) which are true in other interesting contexts.

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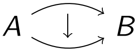
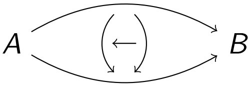
Examples

- All functions $\mathbb{R} \to \mathbb{R}$ are continuous.
- All functions $\mathbb{N} \to \mathbb{N}$ are computable.
- There exist real "infinitesimals" with which we can do calculus (*synthetic differential geometry*).

Outline

Higher categories

A higher category is a category together with "higher morphisms" or "higher homotopies" between its morphisms. Thus it has:

- Objects A, B, ...
- Morphisms $A \to B$,
- 2-morphisms 
- 3-morphisms 
- ...

We will consider only $(\infty, 1)$-categories, where all the higher morphisms are "invertible".

Higher categories

Examples

- Topological spaces
- Simplicial sets

} the "original model", analogous to sets

- Chain complexes
- Categories
- Spectra (stable spaces)
- Simplicial sheaves

Homotopy type theory

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Slogan

Objects of an $(\infty, 1)$-category "look internally" like spaces, a.k.a. homotopy types, a.k.a. ∞-groupoids.

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What we have:

- An almost-complete model in any $(\infty, 1)$-topos (Lumsdaine–Warren and others).
- A complete model in the "standard" case of simplicial sets (Voevodsky).
- Complete models in a small class of presheaf $(\infty, 1)$-toposes (Shulman).

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Question

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Some things we know:

- The law of excluded middle, the axiom of choice, all fail in general (just as for 1-categories).
- "Whitehead's theorem" fails: a homotopy type is not determined by its homotopy groups (Lurie).
- But $\pi_1(S^1) = \mathbb{Z}$ (Shulman).

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Open Problem

Are all the homotopy groups of spheres determined by type theory? Or could they be different in different categories?

Problem 3

Question

What interesting nonclassical axioms can hold in higher categories?

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Example

A cohesive $(\infty, 1)$-topos is one whose objects are homotopy types equipped with some sort of topological or smooth structure. Its internal type-theory admits higher modalities, using which one can describe large chunks of differential cohomology and gauge theory (Schreiber–Shulman).

Thanks!