Cell complexes and inductive definitions

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Homotopy type theory

Homotopy theory

Intensional type theory
Homotopy type theory

Types have a homotopy theory

Homotopy theory

Intensional type theory
Homotopy type theory

Homotopy theory \rightarrow \text{types have a homotopy theory} \rightarrow \text{Intensional type theory}

That means . . .

- Types form a model category (almost) with equivalences, fibrations, cofibrations
- We care about homotopically meaningful constructions
- . . .
Homotopy type theory

Homotopy theory

Intensional type theory

types have a homotopy theory

type theory is a language for homotopy theory
Homotopy type theory

That means...

- Type theory is a formal system, like ZFC
- Homotopy theory can be formalized in type theory
Homotopy type theory

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What is this good for?

- A more direct formalization than in ZFC
- A more computer-friendly formal system than ZFC
- The same proof can apply to many homotopy theories (equivariant, parametrized, sheaves, ...)
Homotopy type theory

Homotopy theory

homotopy (type theory)

(homotopy type) theory

Intensional type theory
Homotopy type theory

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(homotopy type) theory

cell complexes

inductive definitions
Homotopy type theory

Homotopy theory \rightarrow \text{homotopy (type theory)} \rightarrow \text{Intensional type theory} \leftarrow (\text{homotopy type}) \text{ theory}

0-dimensional cell complexes \leftarrow \text{inductive definitions}
Homotopy type theory

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0-dimensional cell complexes

inductive definitions

cell complexes

higher inductive definitions
Outline

1. Inductive definitions
2. Higher inductive definitions
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2. Higher inductive definitions
The natural numbers

Definition
The natural numbers $\mathbb{N}$ are inductively defined by

1. an element $0 \in \mathbb{N}$
2. an operation $s : \mathbb{N} \to \mathbb{N}$.

What does this mean?
The natural numbers

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What does this mean?

Answer #1

- 0 is a natural number;
- for any natural number $x$ there is another called $s(x)$;
- and every natural number is constructed in exactly one of these ways.
The natural numbers

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2. an operation $s : \mathbb{N} \to \mathbb{N}$.

What does this mean?

Answer #2

- $(\mathbb{N}, 0, s)$ is an initial object in the category whose objects are triples $(X, 0_X \in X, s_X : X \to X)$. 
More inductive definitions

Example
For any set $X$, the set $L_X$ of finite lists of elements of $X$ is inductively defined by

1. the empty list $\epsilon \in L_X$
2. the “cons” operation $X \times L_X \rightarrow L_X$. 
More inductive definitions

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Example
The set $T$ of finite binary rooted trees is inductively defined by

1. a leaf node $\ell \in T$
2. a “branch node” operation $T \times T \to T$. 
Definition
An inductive definition of a set $A$ is a list of constructor types

$$F_i(A) \to A$$

where each $F_i$ is an endofunctor of $\mathbf{Set}$.

- For $\mathbb{N}$, we have $F_0(A) = 1$, $F_1(A) = A$.
- For $L_X$, we have $F_0(A) = 1$, $F_1(A) = X \times A$.
- For $T$, we have $F_0(A) = 1$, $F_1(A) = A \times A$.

The set being defined is the initial object of the category of sets equipped with such constructor maps.
Non-recursive inductive definitions

The domains of the constructors don’t have to involve $A$ at all.

Examples

- For any sets $X$ and $Y$, their disjoint union $X \sqcup Y$ is inductively defined by
  - a function $X \rightarrow X \sqcup Y$
  - a function $Y \rightarrow X \sqcup Y$

- A three-element set $Z$ is inductively defined by
  - An element $a \in Z$
  - An element $b \in Z$
  - An element $c \in Z$

- The empty set $\emptyset$ is inductively defined by
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- The empty set \( \emptyset \) is inductively defined by
Theorem
Any inductive definition defines an essentially unique set.

Proof.
Uniqueness is easy (it is initial in some category). For existence, construct $A_0 \to A_1 \to A_2 \to \cdots$ as follows.

1. Let $A_0 = \emptyset$.
2. Let each $A_{n+1}$ be $A_n$ plus new images for all constructors acting on elements of $A_n$ (that haven’t been added yet).

This eventually converges. $\square$
Constructing inductively defined sets

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This eventually converges.

Example

For $\mathbb{N}$, we have $\emptyset \to \{0\} \to \{0, 1\} \to \{0, 1, 2\} \to \cdots$
An inductive definition is a precise description of a universal property for a set, from which we can automatically extract an iterative construction of that set.
Outline

1. Inductive definitions
2. Higher inductive definitions
From sets to spaces

Observation
The iterative construction

\[ A_0 \to A_1 \to A_2 \to \cdots \]

is a 0-dimensional cell complex. The constructors tell us when to glue in a new 0-cell.
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Question
Can we describe more general cell complexes with “inductive definitions”?

An “\(n\)-cell constructor” will need to specify, not just when to glue in a new \(n\)-cell, but what its attaching map should be.
Higher inductive definitions

Definition (Lumsdaine, S.)
A higher inductive definition of a space $A$ is a list of constructor types, each of which has

- a domain $F_i(A)$; and
- a codomain which is one of
  1. $A$;
  2. the space of paths between two specified points of $A$;
  3. the space of homotopies between two specified paths in $A$;
  4. \ldots

Remark
Instead of "homotopies between two specified paths in $A$", we could say "nullhomotopies of a specified loop in $A$", or any other way to describe a 1-sphere in $A$. This way just matches the type theory better.
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Cell complexes as higher induction

\[ \begin{array}{c}
0 \\
\hline
1 \\
\end{array} \]

Example
The interval \( I \) is inductively defined by

1. a point \( 0 \in I \)
2. a point \( 1 \in I \)
3. a path \( 0 \Rightarrow 1 \)

Thus \( I \) is initial in the category of spaces \( X \) equipped with two points \( 0_X, 1_X \in X \) and a path \( 0_X \Rightarrow 1_X \).
Cell complexes as higher induction

Example
The \textit{circle} $S^1$ is inductively defined by
\begin{enumerate}
  \item a point $0 \in S^1$
  \item a path $0 \rightsquigarrow 0$
\end{enumerate}

\textbf{NB:} the path $0 \rightsquigarrow 0$ is not the constant path!
Cell complexes as higher induction

Example
$S^1$ is also inductively defined by

1. two points $0, 1 \in S^1$
2. two paths $0 \rightsquigarrow 1$, $0 \rightsquigarrow 1$
Cell complexes as higher induction

Example
The torus $T^2$ is inductively defined by

1. a point $0 \in T^2$
2. a path $\ell : 0 \rightsquigarrow 0$
3. a path $m : 0 \rightsquigarrow 0$
4. a path $\ell \ast m \rightsquigarrow m \ast \ell$ (where $\ast$ denotes path concatenation)
The **homotopy pushout** of $f: X \to Y$ and $g: X \to Z$ is inductively defined by

1. a map $h: Y \to P$
2. a map $k: Z \to P$
3. for every $x \in X$, a path $h(f(x)) \rightsquigarrow k(g(x))$

**Remark**

Everything happens in the category of spaces, so all constructors are automatically continuous. In particular, $h(f(x)) \rightsquigarrow k(g(x))$ depends continuously on $x$. 
The “very bottom” \( X_{-1} \) is
- empty if \( X \) is empty, and
- contractible if \( X \) is nonempty.

It is inductively defined by
1. a map \( X \to X_{-1} \)
2. for every \( x, y \in X_{-1} \), a path \( x \leadsto y \)

Remark
Again, the path \( x \leadsto y \) depends continuously on \( x \) and \( y \).
Similarly, $X_0$ is inductively defined by

1. a map $X \rightarrow X_0$
2. for any $x, y \in X_0$ and paths $\alpha, \beta : x \rightsquigarrow y$, a path $\alpha \rightsquigarrow \beta$.

$X_1$ is inductively defined by

1. a map $X \rightarrow X_1$
2. for any $x, y \in X_1$, any paths $\alpha, \beta : x \rightsquigarrow y$, and any paths $\mu, \nu : \alpha \rightsquigarrow \beta$, a path $\mu \rightsquigarrow \nu$.

and so on...
Constructing higher inductive types

**Theorem**

*Any higher inductive definition defines an essentially unique space.*

**Proof.**

Proceed as before, constructing $A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots$. This time, at each step we glue in cells of appropriate dimensions corresponding to all the constructors.
Localization

Given $f: A \to B$.

**Definition**

$Z$ is $f$-local if $\text{Map}(B, Z) \xrightarrow{f^*} \text{Map}(A, Z)$ is an equivalence. An $f$-localization of $X$ is an (up-to-homotopy) reflection of $X$ into $f$-local spaces.
**Localization**

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**Definition**

\( Z \) is \( f \)-local if \( \text{Map}(B, Z) \xrightarrow{f^*} \text{Map}(A, Z) \) is an equivalence.

An \( f \)-localization of \( X \) is an (up-to-homotopy) reflection of \( X \) into \( f \)-local spaces.

**First try**

The \( f \)-localization \( X_f \) of \( X \) is inductively defined by

1. a map \( X \to X_f \)
2. for any \( g : A \to X_f \) and \( b \in B \), a point \( e_g(b) \in X_f \)
3. for any \( g : A \to X_f \) and \( a \in A \), a path \( e_g(f(a)) \rightsquigarrow g(a) \)
4. for any \( h : B \to X_f \) and \( b \in B \), a path \( e_{h \circ f}(b) \rightsquigarrow h(b) \)

**Idea:** \( g \mapsto e_g \) defines \( \text{Map}(A, X_f) \to \text{Map}(B, X_f) \), and the two path-constructors make it a homotopy inverse to \( f^* \).
• This space $X_f$ is $f$-local.
• But it is not the $f$-localization of $X$: it is (homotopy) initial among spaces under $X$ equipped with a chosen homotopy inverse to $f^*$.  
• We need “homotopy equivalence data” for $f^*$ which lives in a contractible space.
Localization

Second try (this one works)
The $f$-localization $X_f$ of $X$ is inductively defined by

1. a map $X \to X_f$
2. for $g: A \to X_f$ and $b \in B$, a point $e_g(b) \in X_f$
3. for $g: A \to X_f$ and $a \in A$, a path $\sigma_g(a): e_g(f(a)) \rightsquigarrow g(a)$
4. for $h: B \to X_f$ and $b \in B$, a path $\rho_h(b): e_{h \circ f}(b) \rightsquigarrow h(b)$
5. for $h: B \to X_f$ and $a \in A$, a path $\rho_h(f(a)) \rightsquigarrow \sigma_{h \circ f}(a)$
Third try (this one works too)

The $f$-localization $X_f$ of $X$ is inductively defined by

1. a map $X \to X_f$
2. for $g: A \to X_f$ and $b \in B$, a point $e^1_g(b) \in X_f$
3. for $g: A \to X_f$ and $b \in B$, a point $e^2_g(b) \in X_f$
4. for $g: A \to X_f$ and $a \in A$, a path $\sigma_g(a): e^1_g(f(a)) \leadsto g(a)$
5. for $h: B \to X_f$ and $b \in B$, a path $\rho_h(b): e^2_{h \circ f}(b) \leadsto h(b)$
Spectrification

Definition
A prespectrum is a sequence of pointed spaces \( \{ X_n \mid n \in \mathbb{N} \} \) and maps \( \gamma_n : X_n \to \Omega X_{n+1} \).
It is an (\( \Omega \)-)spectrum if each \( \gamma_n \) is an equivalence.
Spectrification

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A prespectrum is a sequence of pointed spaces \( \{ X_n \mid n \in \mathbb{N} \} \) and maps \( \gamma_n : X_n \to \Omega X_{n+1} \).
It is an \((\Omega-)\)spectrum if each \( \gamma_n \) is an equivalence.

The spectrification \( \{ LX_n \} \) of \( \{ X_n \} \) is inductively defined by

1. maps \( \ell_n : X_n \to LX_n \)
2. for each \( x \in LX_n \), a path \( \ell_{n+1}(*) \sim \ell_{n+1}(*) \)
   (i.e. a map \( L\gamma_n : LX_n \to \Omega(LX_{n+1}) \))
3. for each \( x \in X_n \), a path \( L\gamma_n(\ell_n(x)) \sim (\Omega\ell_{n+1})(\gamma_n(x)) \)
4. data making each \( L\gamma_n \) an equivalence, as for localization.
A higher inductive definition is a precise description of a universal property for a space, from which we can automatically extract an iterative construction of that space.
More Information

http://www.homotopy type theory.org