Traces in indexed monoidal categories

Mike Shulman$^1$  Kate Ponto$^2$

$^1$University of California, San Diego
San Diego, California

$^2$University of Kentucky
Lexington, Kentucky

Octoberfest 2011, Ottawa
October 23, 2011
Outline

1. Indexed monoidal categories
2. String diagrams
3. Traces and fixed point theory
4. Total duality and trace
Indexed monoidal categories

Let $S$ have finite products.

Definition

An $S$-indexed symmetric monoidal category is a pseudofunctor

$$
S^{op} \to \text{SymMonCat}
$$

$$
A \mapsto (\mathcal{C}^A, \otimes_A, I_A)
$$

$$
(A \xrightarrow{f} B) \mapsto (f^*: \mathcal{C}^B \to \mathcal{C}^A)
$$
The external product

Definition
For $M \in \mathcal{C}^A$ and $N \in \mathcal{C}^B$, we set

$$M \boxtimes N := (\pi_1^* M \otimes_{A \times B} \pi_2^* N) \in \mathcal{C}^{A \times B}$$

This is coherently associative and symmetric with unit $I_1$. Moreover, for $M, N \in \mathcal{C}^A$ we have:

$$M \otimes_A N = (\Delta_A)^*(M \boxtimes N).$$
Some indexed monoidal categories

Toy example

$S = \text{sets}$

$\mathcal{C}^A = A$-indexed families of vector spaces (say)
Some indexed monoidal categories

Toy example
\( \mathcal{S} = \text{sets} \)
\( \mathcal{C}^A = A\text{-indexed families of vector spaces (say)} \)

Motivating example
\( \mathcal{S} = \text{topological spaces} \)
\( \mathcal{C}^A = \text{the homotopy category of spectra parametrized over } A \)
Some indexed monoidal categories

Toy example
$\mathcal{S} =$ sets
$\mathcal{C}^A =$ $A$-indexed families of vector spaces (say)

Motivating example
$\mathcal{S} =$ topological spaces
$\mathcal{C}^A =$ the homotopy category of spectra parametrized over $A$

An intermediate example
$\mathcal{S} =$ groupoids
$\mathcal{C}^A =$ the derived category of $A$-diagrams of chain complexes
Some indexed monoidal categories

Toy example
\( S = \text{sets} \)
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\( S = \text{topological spaces} \)
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An intermediate example
\( S = \text{groupoids} \)
\( \mathcal{C}^A = \text{the derived category of } A\text{-diagrams of chain complexes} \)

Uninteresting example (for us)
\( S = \text{any category with pullbacks} \)
\( \mathcal{C}^A = \text{the slice category } S/A \)
**Indexed coproducts**

**Definition**
An indexed category has **indexed coproducts** if each $f^*$ has a left adjoint $f_!$, and any pullback square

\[
\begin{array}{ccc}
A & \xrightarrow{k} & B \\
\downarrow{h} & & \downarrow{f} \\
C & \xrightarrow{g} & D
\end{array}
\]

satisfies the **Beck-Chevalley condition** that

\[
h_! k^* \quad \to \quad h_! k^* f^* f_! \quad \cong \quad h_! h^* g^* f_! \quad \to \quad g^* f_!
\]

is an isomorphism.
Productive pullback squares

The following are always pullback squares:

\[
\begin{array}{ccc}
A \times C & \xrightarrow{f \times \text{id}_C} & B \times C \\
\text{id}_A \times g & \downarrow & \text{id}_B \times g \\
A \times D & \xrightarrow{f \times \text{id}_D} & B \times D \\
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{\Delta} & A \times A \\
\Delta & \downarrow & \text{id}_A \times \Delta \\
A \times A & \xrightarrow{\Delta \times \text{id}_A} & A \times A \times A \\
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{(\text{id}_A, f)} & A \times B \\
f & \downarrow & f \times \text{id}_B \\
B & \xrightarrow{\Delta_B} & B \times B \\
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{\text{id}_A} & A \\
\text{id}_A & \downarrow & \Delta \\
A & \xrightarrow{\Delta} & A \times A \\
\end{array}
\]
Homotopy pullback squares

But: Examples like spectra and chain complexes only satisfy the Beck-Chevalley condition for homotopy pullback squares! (And this is what makes things interesting!)

\[
\begin{align*}
A \times C &\xrightarrow{f \times \text{id}_C} B \times C \\
\text{id}_A \times g &\downarrow \quad \text{id}_B \times g \\
A \times D &\xrightarrow{f \times \text{id}_D} B \times D \\
\text{id}_A \times \Delta &\downarrow \quad \text{id}_A \times \Delta \\
A \times A &\xrightarrow{\Delta \times \text{id}_A} A \times A \times A
\end{align*}
\]
Definition

\( \otimes \) preserves indexed coproducts if the adjunctions \( f_! \dashv f^* \) are all “Hopf adjunctions”, i.e. each map

\[
\begin{align*}
    f_!(M \otimes_A f^* N) & \longrightarrow f_!(f^* f_! M \otimes_A f^* N) \\
    & \cong f_! f^* (f_! M \otimes_B N) \longrightarrow f_! M \otimes_B N
\end{align*}
\]

is an isomorphism.

Lemma

This is equivalent to the canonical morphism

\[
(f \times g)_!(M \boxtimes N) \longrightarrow f_! M \boxtimes g_! N
\]

always being an isomorphism.
Outline

1. Indexed monoidal categories
2. String diagrams
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String diagrams for objects

We start with string diagrams for morphisms in $\mathbf{S}$, proceeding down the page, with morphisms in inverted triangles.

Empty triangles denote

- **diagonal morphisms** $\Delta: A \to A \times A$:

- **projection morphisms** $\pi: A \to 1$:
String diagrams for objects

We add a second type of vertex, labeled by objects of fiber categories.

For $M \in \mathcal{C}^{D \times C}$ and $N \in \mathcal{C}^{D \times B}$, this diagram denotes

$$(g \times h \times \text{id}_B)^* (M \boxtimes N)$$
We add a second type of vertex, labeled by objects of fiber categories.

For $M \in \mathcal{C}^{D \times C}$ and $N \in \mathcal{C}^{D \times B}$, this diagram denotes

\[(g \times h \times \text{id}_B)^*(M \boxtimes N)\quad \text{or}\quad (h \times \text{id})^*(g \times \text{id})^*(M \boxtimes N)\]
String diagrams for objects

We add a second type of vertex, labeled by objects of fiber categories.

For $M \in \mathcal{C}^{D \times C}$ and $N \in \mathcal{C}^{D \times B}$, this diagram denotes

$$(g \times h \times \text{id}_B)^*(M \boxtimes N) \quad \text{or} \quad (h \times \text{id})^*(g \times \text{id})^*(M \boxtimes N) \quad \text{or} \quad (h \times \text{id})^*((g \times \text{id})^*M) \boxtimes N).$$
Indexed coproducts

We add a third type of vertex, labeled by morphisms of $\mathbf{S}$ inside upward-pointing triangles, with source and target reversed.

For $f: A \to B$, $g: C \to A$, $M \in \mathcal{C}^A$, $N \in \mathcal{C}^C$, and $P \in \mathcal{C}^D$, this diagram denotes

$$(f_! \Delta^*(M \boxtimes g_! N)) \boxtimes \pi_! P \quad \in \mathcal{C}^B$$
Productive pullback squares

The following are always pullback squares:

\[
\begin{align*}
A \times C \xrightarrow{f \times \text{id}_C} B \times C \\
\text{id}_A \times g \downarrow \quad \downarrow \text{id}_B \times g \\
A \times D \xrightarrow{f \times \text{id}_D} B \times D
\end{align*}
\]

\[
\begin{align*}
A \xrightarrow{\Delta} A \times A \\
\Delta \downarrow \quad \downarrow \text{id}_A \times \Delta \\
A \times A \xrightarrow{\Delta \times \text{id}_A} A \times A \times A
\end{align*}
\]

\[
\begin{align*}
A \xrightarrow{(\text{id}_A, f)} A \times B \\
f \downarrow \quad \downarrow f \times \text{id}_B \\
B \xrightarrow{\Delta_B} B \times B
\end{align*}
\]

\[
\begin{align*}
A \xrightarrow{\text{id}_A} A \\
\text{id}_A \downarrow \quad \downarrow \Delta \\
A \xrightarrow{\Delta} A \times A
\end{align*}
\]
Beck-Chevalley conditions

The corresponding Beck-Chevalley conditions say:
⊗ preserves indexed coproducts

Recall that ⊗ preserves indexed coproducts if and only if

\[(f \times g)!(M \boxtimes N) \cong f!M \boxtimes g!N.\]

This is necessary in order to interpret the following diagram unambiguously:
First we turn the string diagrams for objects on their sides...

(Think of the right-hand diagram as written on a piece of paper lying horizontally perpendicular to the screen.)
String diagrams for morphisms

... then we add nodes between these “horizontal slices”, with strings connecting them to the nodes they affect.

This depicts

\[ f^* h^* M \xrightarrow{f^*(\phi)} f^* g^* N \xrightarrow{\cong} (gf)^* N \]
Example: naturality

For $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{S}$ and $\phi: M \rightarrow N$ in $\mathcal{C}^D$, the following square commutes:

$$
\begin{array}{ccc}
\begin{array}{c}
f^*g^* M \\
\downarrow f^*g^* \phi
\end{array} & \xrightarrow{\cong} & \begin{array}{c}
(gf)^* M \\
\downarrow (gf)^* \phi
\end{array} \\
\downarrow & & \downarrow \\
\begin{array}{c}
f^*g^* N \\
\cong
\end{array} & \xrightarrow{\cong} & \begin{array}{c}
(gf)^* N
\end{array}
\end{array}
$$

by naturality of $f^*g^* \cong (gf)^*$.
Example: naturality

Equivalently:
A triangle identity for $g! \dashv g^*$
A triangle identity for $\Delta_1 \vdash \Delta^*$
Beck-Chevalley conditions, again
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Classical fixed point theory in a nutshell

Definition
Let \( C \) be a symmetric monoidal category, \( M \in C \) a dualizable object, and \( f : M \to M \). The trace of \( f \) is the composite

\[
I \xrightarrow{\eta} M \otimes M^\ast \xrightarrow{f \otimes 1} M \otimes M^\ast \xrightarrow{\simeq} M^\ast \otimes M \xrightarrow{\varepsilon} I.
\]

Paradigm
Let \( S \) be cartesian monoidal, \( C \) symmetric monoidal and “additive”, and \( \Sigma : S \to C \) a symmetric monoidal functor. Given \( f : A \to A \) in \( S \), if \( \Sigma(A) \) is dualizable, then the trace of \( \Sigma(f) \) gives information about the fixed points of \( f \).
A larger nutshell

For any \( \mathcal{S} \)-indexed symmetric monoidal category \( \mathcal{C} \), we have a symmetric monoidal functor

\[
\Sigma: (\mathcal{S}, \times, 1) \to (\mathcal{C}^1, \otimes_1, I_1)
\]

\[
A \mapsto (\pi_A)! I_A \cong (\pi_A)! (\pi_A)^* I_1
\]

Most interesting examples of \( \Sigma \) seem to arise in this way.
The toy example

- \( S = \text{sets} \)
- \( \mathcal{C}^1 = \text{vector spaces} \)
- \( \Sigma(A) = \text{a vector space with basis } A \)
- \( A \text{ finite } \Rightarrow \Sigma(A) \text{ dualizable} \)
- \( \Sigma(f) = \text{the matrix with entries } m_{a,f(a)} = 1 \text{ and others } 0 \)
- \( \text{tr}(\Sigma(f)) = \sum_{a \in A} m_{a,a} = \text{the number of fixed points of } f. \)
(Dold–Puppe)

- \( S = \) topological spaces
- \( \mathcal{C}^1 = \) the stable homotopy category of spectra

- \( \Sigma(A) = \) the suspension spectrum \( \Sigma_\infty(A) \)
- \( A \) a closed smooth manifold \( \Rightarrow \Sigma(A) \) dualizable

- \( \text{tr}(\Sigma(f)) = \) the total fixed-point index of \( f \)
The motivating example

(Dold–Puppe)

- $S =$ topological spaces
- $\mathcal{C}^1 =$ the stable homotopy category of spectra

- $\Sigma(A) =$ the suspension spectrum $\Sigma_+^\infty(A)$
- $A$ a closed smooth manifold $\Rightarrow \Sigma(A)$ dualizable

- $\text{tr}(\Sigma(f)) =$ the total fixed-point index of $f$

Remarks

- The “homology” functor on spectra preserves traces, so $\text{tr}(\Sigma(f)) = \text{tr}(H_*(f))$, the Lefschetz number.
- If $f$ has no fixed points, $\text{tr}(\Sigma(f)) = 0$. This is the Lefschetz fixed-point theorem.
The intermediate example

- $\mathcal{S} = \text{groupoids}$
- $\mathcal{C}^1 = \text{the derived category of chain complexes}$
- $\Sigma(A) = \text{simplicial chains on the nerve of } A$
- $A$ finitely generated and free $\Rightarrow \Sigma(A) \text{ dualizable}$

$\text{tr}(\Sigma(f)) =$ the sum of:
- the number of objects of $A$ (literally) fixed by $f$;
- for each generating morphism $\gamma$ of $A$,
  - the number of times $\gamma^{-1}$ appears in $f(\gamma)$,
  - *minus* the number of times $\gamma$ appears in $f(\gamma)$. 
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Constructing a bicategory

Theorem (May–Sigurdsson, S.)

There is a bicategory $\mathcal{C}$ with

- objects of $\mathcal{C}$ = objects of $\mathcal{S}$
- hom-categories $\mathcal{C}(A, B) = \mathcal{C}^{A \times B}$

$\mathcal{C}$ is symmetric monoidal and every object is its own dual.

Examples

- $\mathcal{S} =$ sets, $\mathcal{C} =$ families of vector spaces
  $\Rightarrow \mathcal{C} =$ matrices of vector spaces
- $\mathcal{S} =$ groupoids, $\mathcal{C} =$ diagrams of chain complexes
  $\Rightarrow \mathcal{C} =$ chain-complex-valued profunctors
The bicategory structure

\[ M \oslash N = (\pi_{13})^! \Delta^*(M \boxtimes N) \]

\[ U_A = \Delta^! \pi_A^* l_1 \]
Associativity

$(M \odot N) \odot P$
Associativity
$M \odot (N \odot P)$
Unitality

$M \otimes U_B$
Unitality
Unitality
Embedding $\mathcal{C}$ in $\mathcal{C}$

We have equivalences

$$\mathcal{C}^A \simeq \mathcal{C}(A, 1) \simeq \mathcal{C}(1, A)$$

$M \mapsto \hat{M} \mapsto \hat{M}$
Embedding $\mathcal{C}$ in $\mathcal{C}$

We have equivalences

$$\mathcal{C}^A \simeq \mathcal{C}(A, 1) \simeq \mathcal{C}(1, A)$$

$M \mapsto \hat{M} \mapsto \tilde{M}$

Theorem (May–Sigurdsson, Ponto–S.)

$\hat{M}$ has an adjoint in $\mathcal{C}$ $\iff$ $M$ is dualizable in $\mathcal{C}^A$

Proof.
By string diagram manipulation...
Total duality

Definition
$M \in C^A$ is totally dualizable if $\tilde{M}$ has an adjoint in $C$.

We say $A \in S$ is totally dualizable if $I_A$ is.
Total duality

Definition

$M \in \mathcal{C}^A$ is **totally dualizable** if $\hat{M}$ has an adjoint in $\mathcal{C}$.

We say $A \in \mathcal{S}$ is totally dualizable if $I_A$ is.

**Theorem (May–Sigurdsson)**

*If $A$ is totally dualizable, then $\Sigma(A)$ is dualizable in $\mathcal{C}^1$.***

**Proof.**

- $\Sigma(A) \cong \hat{I}_A \odot \hat{I}_A$ (check).
- But $\hat{I}_A$ always has an adjoint, since $I_A \in \mathcal{C}^A$ is always dualizable.

□
Total duality

Definition
$M \in \mathcal{C}^A$ is **totally dualizable** if $\tilde{M}$ has an adjoint in $\mathcal{C}$.

We say $A \in \mathcal{S}$ is totally dualizable if $I_A$ is.

Theorem (May–Sigurdsson)
*If $A$ is totally dualizable, then $\Sigma(A)$ is dualizable in $\mathcal{C}^1$.*

Proof.
- $\Sigma(A) \cong \tilde{I}_A \odot \tilde{I}_A$ (check).
- But $\tilde{I}_A$ always has an adjoint, since $I_A \in \mathcal{C}^A$ is always dualizable.

The converse usually holds in practice.
Problem
Suppose $M: A \to B$ is a 1-cell in a bicategory $\mathcal{B}$ with an adjoint $M^\star$, and $f: M \to M$. For its trace, we want

$$U_A \xrightarrow{\eta} M \otimes M^\star \xrightarrow{f \otimes 1} M \otimes M^\star \quad ??? \quad M^\star \otimes M \xrightarrow{\varepsilon} U_B.$$ 

But $M \otimes M^\star: A \to A$ while $M^\star \otimes M: B \to B$. 

Traces in bicategories
Problem
Suppose $M : A \to B$ is a 1-cell in a bicategory $\mathcal{B}$ with an adjoint $M^\star$, and $f : M \to M$. For its trace, we want

$$U_A \xrightarrow{\eta} M \circ M^\star \xrightarrow{f \otimes 1} M \circ M^\star \quad ??? \quad M^\star \circ M \xrightarrow{\varepsilon} U_B.$$  

But $M \circ M^\star : A \to A$ while $M^\star \circ M : B \to B$.

Solution (Ponto)
If $\mathcal{B}$ is symmetric monoidal and $A$ and $B$ have duals, then endo-1-cells have traces, which are cyclic. We call

$$\text{Tr}(U_A) \xrightarrow{\eta} \text{Tr}(M \circ M^\star) \xrightarrow{f \otimes 1} \text{Tr}(M \circ M^\star) \xrightarrow{\cong} \text{Tr}(M^\star \circ M) \xrightarrow{\varepsilon} \text{Tr}(U_B).$$

the trace of $f$. 

Traces in bicategories
Total-duality traces

If $A$ is totally dualizable with dual $N$, the total-duality trace of $f : A \to A$ is the twisted trace of $I_A \cong I_A \circ A_f$:

$$I_1 \xrightarrow{\eta} \text{Tr}(\tilde{I}_A \circ N) \xrightarrow{\cong} \text{Tr}(\tilde{I}_A \circ A_f \circ N) \xrightarrow{\cong} \text{Tr}(A_f \circ N \circ \tilde{I}_A) \xrightarrow{\varepsilon} \text{Tr}(A_f).$$

Here $A_f = (1 \times f)^*(\Delta_A)! I_A$. 

Theorem (Ponto–S.)

The trace of $\Sigma(f)$ factors through the total-duality trace of $f$:

Proof. An elementary property of traces.
Total-duality traces

If $A$ is totally dualizable with dual $N$, the total-duality trace of $f: A \to A$ is the twisted trace of $I_A \cong I_A \circ A_f$:

$$I_1 \xrightarrow{\eta} \text{Tr}(I_A \circ N) \xrightarrow{\cong} \text{Tr}(I_A \circ A_f \circ N) \xrightarrow{\cong} \text{Tr}(A_f \circ N \circ I_A) \xrightarrow{\epsilon} \text{Tr}(A_f).$$

Here $A_f = (1 \times f)^*(\Delta_A)!I_A$.

**Theorem (Ponto–S.)**

The trace of $\Sigma(f)$ factors through the total-duality trace of $f$:

$$I_1 \xrightarrow{\text{Tr}(A_f)} \text{Tr}(A_f) \xrightarrow{\text{tr}(\Sigma(f))} I_1.$$

**Proof.**

An elementary property of traces.
The toy example

- \( S = \text{sets, } \mathcal{C}^A = A\)-indexed families of vector spaces.
- \( \text{Tr}(A_f) \) = a vector space with basis \( \text{Fix}(f) \), the set of fixed points of \( f \).
- The map \( \text{Tr}(A_f) \to \mathbb{k} \) sends each basis element to 1.
- The total-duality trace of \( f \) is the formal sum of all the fixed points of \( f \).

\[
\#\{x \mid f(x) = x\}.
\]
The motivating example

- **S** = spaces, \( \mathcal{C}^A \) = spectra parametrized over \( A \).
- Total duality is called **Costenoble–Waner duality**.
- \( \text{Tr}(A_f) \) = the suspension spectrum of the twisted free loop space

\[
\Lambda^f A = \left\{ \gamma : [0, 1] \to A \mid \gamma(1) = f(\gamma(0)) \right\}
\]

- The total-duality trace of \( f \) is the **Reidemeister trace**. The theorem says that it refines the fixed-point index.
The motivating example

- $S =$ spaces, $\mathcal{C}^A =$ spectra parametrized over $A$.
- Total duality is called Costenoble–Waner duality.
- $\text{Tr}(A_f) =$ the suspension spectrum of the twisted free loop space

$$\Lambda^f A = \{ \gamma : [0, 1] \to A \mid \gamma(1) = f(\gamma(0)) \}$$

- The total-duality trace of $f$ is the Reidemeister trace. The theorem says that it refines the fixed-point index.

Theorem (Husseini, Crabb–James)

*If $A$ is a closed manifold of dimension $\geq 3$, then the Reidemeister trace of $f$ vanishes if and only if $f$ is homotopic to a map without fixed points.*
The intermediate example

- $S = \text{groupoids, } C^A = A$-diagrams of chain complexes
- $\text{Tr}(A_f) = \text{simplicial chains on the nerve of the twisted free loop groupoid } \Lambda^f A$, whose objects are morphisms $\gamma: x \to f(x)$ in $A$. Its $H_0$ is the free abelian group on the connected components of $\Lambda^f A$.
- The total-duality trace of $f$ is the sum of the following.
  - For each object $x \in A$ literally fixed by $f$, we add $[\text{id}_x]$.
  - For each generating morphism $\gamma$ and every occurrence of $\gamma^{-1}$ in $f(\gamma)$, say $f(\gamma) = \beta \gamma^{-1} \alpha$, we add $[\beta \gamma^{-1}]$.
  - For each generating morphism $\gamma$ and every occurrence of $\gamma$ in $f(\gamma)$, say $f(\gamma) = \beta \gamma \alpha$, we subtract $[\beta]$. 
What is the categorical meaning of these traces for finitely generated free groupoids?

Do they say anything about “fixed points” in any categorical sense?