On modeling homotopy type theory in higher toposes

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Here we go

Theorem

Every Grothendieck $(\infty, 1)$-topos can be presented by a model category that interprets "Book" Homotopy Type Theory with:

- *Σ-types, a unit type, Π-types with function extensionality, and identity types.*
- *Strict universes, closed under all the above type formers, and satisfying univalence and the propositional resizing axiom.*

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Some caveats

1. Classical metatheory: ZFC with inaccessible cardinals.
2. Classical homotopy theory: simplicial sets. (It's not clear which cubical sets can even model the $(\infty, 1)$-topos of ∞-groupoids.)
3. Will not mention "elementary $(\infty, 1)$-toposes" (though we can deduce partial results about them by Yoneda embedding).
4. Not the full "internal language hypothesis" that some "homotopy theory of type theories" is equivalent to the homotopy theory of some kind of $(\infty, 1)$-category.
   Only a unidirectional interpretation — in the useful direction!
5. We assume the initiality hypothesis: a "model of type theory" means a CwF.

Outline

1. Review: model categories for type theory

2. Left exact localizations

3. Injective fibrations

Fibrations as types

Standard categorical semantics of dependent type theory is:

- A category whose objects represent "types" or "contexts".
- A class of "display maps" $B \twoheadrightarrow A$ representing dependent types $x : A \vdash B(x)$ type.
- Sections of a display map represent terms $x : A \vdash b(x) : B(x)$.
- Further structure corresponding to all the rules of type theory.

Theorem (Awodey–Warren)

The elimination rule for identity types says exactly that the reflexivity term $A \to \mathsf{Id}_A$ has the left lifting property with respect to the display maps. Thus, if we regard display maps as fibrations, then identity types are path objects.

Type-theoretic model categories

Any Quillen model category $\mathscr{E}$ models type theory with fibrations as display maps. The question is which additional rules it also models.

- $\mathscr{E}$ always has a unit type and Σ-types (fibrations contain the identities and are closed under composition).
- If $\mathscr{E}$ is locally cartesian closed, and for any fibration f the dependent product f_* preserves fibrations and acyclic fibrations, then $\mathscr{E}$ has Π-types satisfying function extensionality.
  - Equivalent to f^* preserving acyclic cofibrations and cofibrations.
  - Since f^* always preserves acyclic fibrations, this is equivalent to it preserving weak equivalences and cofibrations.
  - Hence it follows if $\mathscr{E}$ is right proper and cofibration = mono.
- If f^* preserves acyclic cofibrations for any fibration f, then $\mathscr{E}$ has identity types (they have to be pullback-stable).

We also need a coherence theorem, e.g. Lumsdaine–Warren local universes, or Voevodsky universes.

Higher inductive types

Theorem (Lumsdaine–S.)

If $\mathscr{E}$ is right proper, combinatorial, simplicial, simplicially locally cartesian closed, and its cofibrations are the monomorphisms, then it also has most higher inductive types.

"Build them 1-categorically,
mixing in fibrant replacement algebraically."

Everything but universes

Theorem (Summary)

Every locally cartesian closed, locally presentable $(\infty, 1)$-category admits the structure of a model of Homotopy Type Theory with:

- *Σ-types, a unit type, Π-types with function extensionality, and identity types.*
- *Most higher inductive types.*

Proof.

Gepner–Kock presented any such $(\infty, 1)$-category as a semi-left-exact localization of an injective model structure on simplicial presheaves, which has all these properties. □

What about universes?

- Say a morphism is κ-small if it has κ-small fibers.
- Every Grothendieck $(\infty, 1)$-topos has object classifiers for κ-small morphisms, which satisfy univalence (Rezk, Lurie, Gepner–Kock).
- If κ is inaccessible, the κ-small morphisms are closed under everything. . . at the $(\infty, 1)$-category level.
- But in type theory, if $A : \mathcal{U}$ and $B : A \to \mathcal{U}$ then $\prod_{x:A} B(x)$ must literally be an element of $\mathcal{U}$, not just equivalent to one. Coherence theorems can weaken this to isomorphism in models, but not (yet) to equivalence.
- We need a κ-small fibration $\pi : \widetilde{U} \to U$ such that every κ-small fibration is a 1-categorical pullback of π.

Universes in presheaves

Definition

If $\mathscr{E} = [\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$ is a presheaf category, define a presheaf U where $U(c)$ is the "set" of κ-small fibrations over the representable $\mathbf{y}_c = \mathscr{C}(-, c)$. For $\gamma : c_1 \to c_2$, the functorial action $U(c_2) \to U(c_1)$ is by pullback along $\mathbf{y}_\gamma : \mathbf{y}_{c_1} \to \mathbf{y}_{c_2}$.

This takes a bit of work to make precise:

- $U(c)$ must be a set containing at least one representative for each isomorphism class of such κ-small fibrations,
- Chosen cleverly to make pullback strictly functorial.
  Will not discuss this today, nothing is really new here.

Universes in presheaves, II

Similarly, we can define $\widetilde{U}$ to consist of κ-small fibrations equipped with a section. We have a κ-small projection $\pi : \widetilde{U} \to U$.

Theorem

Every κ-small fibration is a pullback of π.

But π may not itself be a fibration! All we can say is that its pullback along any map $x : \mathbf{y}_c \to U$, with $\mathbf{y}_c$ representable, is a fibration (namely the fibration that "is" $x \in U(c)$).

Universes in presheaves, III

Theorem

If the generating acyclic cofibrations in $\mathscr{E} = [\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$ *have representable codomains, then* $\pi : \widetilde{U} \to U$ *is a fibration.*

Proof.

To lift in the left square, we can instead lift in the right square, whose right map is a fibration.

$$\begin{array}{ccc} A & \longrightarrow & \widetilde{U} \\ {\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbf{y}c & \longrightarrow & U \end{array} \qquad \begin{array}{ccc} A & \longrightarrow & \mathbf{y}c \times_U \widetilde{U} \\ {\scriptstyle\sim}\downarrow & & \downarrow \\ \mathbf{y}c & = & \mathbf{y}c \end{array}$$

□

Presheaf universes that do exist

There's more to prove (U is fibrant, and univalence holds), but this is the crux of the matter. Thus we get known examples:

Example

In simplicial sets, the generating acyclic cofibrations are $\Lambda^k[n] \to \Delta[n]$, where $\Delta[n]$ is representable.

Example

In simplicial presheaves on an inverse (or elegant Reedy) category $\mathcal{R}$, the generating acyclic cofibrations are

$$(\Delta[n] \times \partial\mathcal{R}(-,x)) \cup (\Lambda^k[n] \times \mathcal{R}(-,x)) \to \Delta[n] \times \mathcal{R}(-,x)$$

and $\Delta[n] \times \mathcal{R}(-,x)$ is representable by $(n,x) \in \mathbb{\Delta} \times \mathcal{R}$.

The Plan

- A Grothendieck $(\infty, 1)$-topos is, by definition, an accessible left exact localization of a presheaf $(\infty, 1)$-category.
- It can therefore be presented by an accessible left exact localization of an injective model structure on simplicial presheaves, which we know models all of homotopy type theory except universes.

Thus it will suffice to:

1. Understand injective model structures.
2. Understand left exact localizations.

We take these in reverse order.

Outline

Accessible lex modalities

Definition (in HoTT)

A reflective subuniverse consists of

- A predicate $\mathsf{in}_\lozenge : \mathcal{U} \to \mathsf{Prop}$. If $\mathsf{in}_\lozenge(X)$ we say X is modal.
- A reflector $\lozenge : \mathcal{U} \to \mathcal{U}$ with units $\eta_X : X \to \lozenge X$.
- For all X, the type $\lozenge X$ is modal.
- If Y is modal, then $(- \circ \eta_X) : (\lozenge X \to Y) \to (X \to Y)$ is an equivalence.

It is a lex modality if in addition

- $\lozenge$ preserves pullbacks.

A lex modality is accessible if

- There exists $B : A \to \mathcal{U}$ such that X is modal if and only if for all $a : A$ the map $X \to (B_a \to X)$ is an equivalence.

Modal type operations

Theorem (in HoTT)

If $\lozenge$ is an accessible lex modality, then:

1. *If A is modal and $x, y : A$, then $x = y$ is modal.*
2. *If A is modal, and $B : A \to \mathcal{U}$ is such that each $B(x)$ is modal, then $\sum_{x:A} B(x)$ and $\prod_{x:A} B(x)$ are modal.*
3. *The universe of modal types $\sum_{X:\mathcal{U}} \mathrm{in}_{\lozenge}(X)$ is modal.*
4. *We can construct "modal higher inductive types" by adding a nullification constructor to the others of a given HIT.*

Modeling type theory with lex modalities

Theorem

If $\mathscr{E}$ is a model category that interprets homotopy type theory (with universes), and $\lozenge$ is an accessible lex modality in the internal type theory of $\mathscr{E}$, then there is a corresponding localization of $\mathscr{E}$ that also interprets homotopy type theory.

We essentially knew this already back at the IAS special year.

The subtle problem is that it's not clear that every external accessible left exact localization of $\mathscr{E}$ induces an internal accessible lex modality!

Fiberwise reflections

Because a reflective subuniverse is a map $\lozenge : \mathcal{U} \to \mathcal{U}$, it induces a reflective subcategory not just of $\mathscr{E}$ but of all slice categories of $\mathscr{E}$:

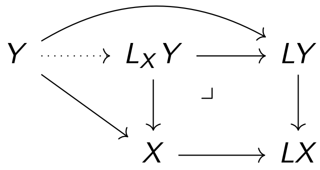
$$\begin{array}{ccc} Y & \longrightarrow & \widetilde{U} \\ \downarrow & \lrcorner & \downarrow \pi \\ X & \longrightarrow & U \end{array} \quad \rightsquigarrow \quad \begin{array}{ccccc} \lozenge_X Y & & \longrightarrow & & \widetilde{U} \\ \downarrow & \lrcorner & & & \downarrow \pi \\ X & \longrightarrow & U & \xrightarrow[\lozenge]{} & U \end{array}$$

If $\lozenge$ is a lex modality, then each $\lozenge_X$ is a left exact reflection.

From localizations to modalities

If $L : \mathscr{E} \to \mathscr{E}$ is an accessible left exact localization at some set of maps S, there are two ways to try to extend it to slice categories:

1. Localize $\mathscr{E}/X$ at the pullback class $X^*(S)$. This yields an accessible reflective subcategory of $\mathscr{E}/X$, but it is not clear that it is left exact.
2. Construct the "reflective factorization system":



This yields a left exact reflective subcategory of $\mathscr{E}/X$, but it is not clear that this family of reflective subcategories is accessible in the internal sense.

Pulling back left exactness

Theorem (Anel–Biedermann–Finster–Joyal, 2019)

If S is closed under diagonals and pullbacks to a generating set, then S-localization is left exact. Every left exact localization can be obtained from such an S.

This property of S is preserved by pullback to X. Thus localizing $\mathscr{E}/X$ at $X^*(S)$ is left exact, hence a lex modality.

Theorem

If $\mathscr{E}$ interprets homotopy type theory (with universes), then so does any accessible left exact localization of it.

Outline

Injective model structures

$\mathscr{S}$ = simplicial sets, $\mathscr{D}$ = a small simplicially enriched category.

Theorem

The category $[\mathscr{D}^{\mathrm{op}}, \mathscr{S}]$ of simplicially enriched presheaves has an injective model structure such that:

1. *The weak equivalences are pointwise.*
2. *The cofibrations are pointwise, hence are the monomorphisms in $[\mathscr{D}^{\mathrm{op}}, \mathscr{S}]$.*
3. *It is right proper, combinatorial, simplicial, and simplicially locally cartesian closed.*
4. *It presents the $(\infty, 1)$-category of $(\infty, 1)$-presheaves on the small $(\infty, 1)$-category presented by $\mathscr{D}$.*

So it models everything but universes.

Injective model structures

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2. *The cofibrations are pointwise, hence are the monomorphisms in $[\mathscr{D}^{\mathrm{op}}, \mathscr{S}]$.*
- *The fibrations are . . . ?????*
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Why pointwise isn't enough

Let $\mathscr{D}$ be unenriched for simplicity. When is $X \in [\mathscr{D}^{\mathrm{op}}, \mathscr{S}]$ injectively fibrant? We want to lift in

$$\begin{array}{ccc} A & \xrightarrow{g} & X \\ {\scriptstyle i}\big\downarrow{\scriptstyle \sim} & \nearrow & \\ B & & \end{array}$$

where $i : A \to B$ is a pointwise acyclic cofibration.
If X is pointwise fibrant, then for all $d \in \mathscr{D}$ we have a lift

$$\begin{array}{ccc} A_d & \xrightarrow{g_d} & X_d \\ {\scriptstyle i_d}\big\downarrow{\scriptstyle \sim} & \nearrow_{h_d} & \\ B_d & & \end{array}$$

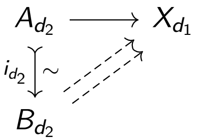
but these may not fit together into a natural transformation $B \to X$.

Naturality up to homotopy

Naturality would mean that for any $\delta : d_1 \to d_2$ in $\mathscr{D}$ we have $X_\delta \circ h_{d_2} = h_{d_1} \circ B_\delta$. This may not hold, but we do have

$$X_\delta \circ h_{d_2} \circ i_{d_2} = X_\delta \circ g_{d_2} = g_{d_1} \circ A_\delta = h_{d_1} \circ i_{d_1} \circ A_\delta = h_{d_1} \circ B_\delta \circ i_{d_2}.$$

Thus, $X_\delta \circ h_{d_2}$ and $h_{d_1} \circ B_\delta$ are both lifts in the following:

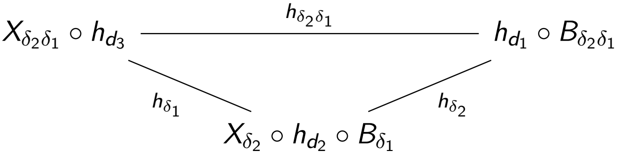


Since lifts between acyclic cofibrations and fibrations are unique up to homotopy, we do have a homotopy

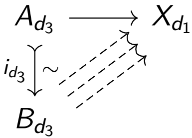
$$h_\delta : X_\delta \circ h_{d_2} \sim h_{d_1} \circ B_\delta.$$

Coherent naturality

Similarly, given $d_1 \xrightarrow{\delta_1} d_2 \xrightarrow{\delta_2} d_3$, we have a triangle of homotopies



whose vertices are lifts in the following:



Thus, homotopy uniqueness of lifts gives us a 2-simplex filler.

Homotopy coherent natural transformations

For $X, Y \in [\mathscr{D}^{\mathrm{op}}, \mathscr{S}]$, a homotopy coherent natural transformation $h : X \rightsquigarrow Y$ consists of:

- For every $d \in \mathscr{D}$, a morphism $h_d : X_d \to Y_d$.
- For every $d_1 \xrightarrow{\delta} d_2$ in $\mathscr{D}$, a homotopy $h_\delta : \Delta[1] \to \mathscr{E}(X_{d_2}, Y_{d_1})$ between $Y_\delta \circ h_{d_2}$ and $h_{d_1} \circ X_\delta$, such that h_{id_d} is constant.
- For every $d_1 \xrightarrow{\delta_1} d_2 \xrightarrow{\delta_2} d_3$ in $\mathscr{D}$, a 2-simplex $h_{\delta_1,\delta_2} : \Delta[2] \to \mathscr{E}(X_{d_3}, Y_{d_1})$ whose boundaries involve h_{δ_1}, h_{δ_2}, and $h_{\delta_2\delta_1}$, satisfying similar constancy conditions.
- And so on.

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- For every d_2 in $\mathscr{D}$, a morphism $X_{d_2} \to \prod_{\delta : d_1 \to d_2} Y_{d_1}^{\Delta[1]}$ between $Y_\delta \circ h_{d_2}$ and $h_{d_1} \circ X_\delta$, such that h_{id_d} is constant.
- For every d_3 in $\mathscr{D}$, a morphism $X_{d_3} \to \prod_{d_1 \xrightarrow{\delta_1} d_2 \xrightarrow{\delta_2} d_3} Y_{d_1}^{\Delta[2]}$ with suitable conditions.
- And so on.

Homotopy coherent natural transformations

For $X, Y \in [\mathscr{D}^{\mathrm{op}}, \mathscr{S}]$, a homotopy coherent natural transformation $h : X \rightsquigarrow Y$ consists of:

- A natural transformation $X \to G(Y_?)$.
- A natural transformation $X \to G\left(\prod_{\delta : d_1 \to ?} Y_{d_1}^{\Delta[1]}\right)$ with suitable conditions.
- A natural transformation $X \to G\left(\prod_{d_1 \xrightarrow{\delta_1} d_2 \xrightarrow{\delta_2} ?} Y_{d_1}^{\Delta[2]}\right)$ with suitable conditions.
- And so on.

Where G is right adjoint to the forgetful $U : [\mathscr{D}^{\mathrm{op}}, \mathscr{S}] \to \mathscr{S}^{\mathrm{ob}\,\mathscr{D}}$ (G is the "cofree presheaf" on a family of objects).

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In fact $(GZ)_{d_1} = \prod_{\delta : d_1 \to d_2} Z_{d_2}$.

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- A natural transformation $X \to GUGUY^{\Delta[1]}$ with suitable conditions.
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Homotopy coherent natural transformations

For $X, Y \in [\mathscr{D}^{\mathrm{op}}, \mathscr{S}]$, a homotopy coherent natural transformation $h : X \rightsquigarrow Y$ consists of:

- A natural transformation $X \to \mathsf{C}^{\mathscr{D}}(Y)$.

Where G is right adjoint to the forgetful $U : [\mathscr{D}^{\mathrm{op}}, \mathscr{S}] \to \mathscr{S}^{\mathsf{ob}\,\mathscr{D}}$ (G is the "cofree presheaf" on a family of objects).
and the cobar construction $\mathsf{C}^{\mathscr{D}}(Y)$ is the totalization of the cosimplicial object

$$GU \rightrightarrows GUGUY \rightrightarrows GUGUGUY \quad \cdots$$

The coherent morphism coclassifier

Conclusion

$\mathsf{C}^{\mathscr{D}}(Y)$ is the coclassifier of coherent transformations: we have a natural bijection

$$\frac{h : X \rightsquigarrow Y}{\overline{h} : X \to \mathsf{C}^{\mathscr{D}}(Y)}$$

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Some more facts:

- The (strictly natural) identity $X \rightsquigarrow X$ corresponds to a canonical map $\nu_X : X \to \mathsf{C}^{\mathscr{D}}(X)$.
- If $h : X \to Y$ is strict, then $\overline{h} : X \to \mathsf{C}^{\mathscr{D}}(Y)$ factors as $\overline{h} = \nu_Y \circ h$.
- ν_X is always a pointwise acyclic cofibration!

Injective fibrancy

Theorem

$X \in [\mathscr{D}^{\mathrm{op}}, \mathscr{S}]$ is injectively fibrant if and only if it is pointwise fibrant and $\nu_X : X \to \mathsf{C}^{\mathscr{D}}(X)$ has a retraction $r : \mathsf{C}^{\mathscr{D}}(X) \to X$.

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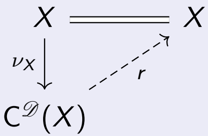
Proof of "only if".

If $X \in [\mathscr{D}^{\mathrm{op}}, \mathscr{S}]$ is injectively fibrant, then since ν_X is a pointwise acyclic cofibration we have a lift:



□

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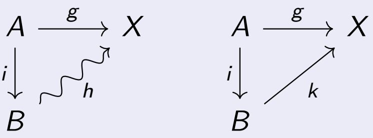
Proof of "if".

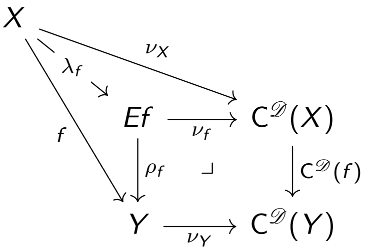
Given a pointwise acyclic cofibration $i : A \to B$ and a map $g : A \to X$, we construct a coherent $h : B \rightsquigarrow X$ with $h \circ i = g$.



We have $\overline{h} : B \to \mathsf{C}^{\mathscr{D}}(X)$; define $k = r \circ \overline{h} : B \to X$. Since $h \circ i = g$ is strict, $\overline{h} \circ i = \nu_X \circ g$, and $k \circ i = r \circ \overline{h} \circ i = r \circ \nu_X \circ g = g$. □

Injective fibrations

Given $f : X \to Y$, define a factorization by pullback:



Theorem

$f : X \to Y$ is an injective fibration if and only if it is a pointwise fibration and λ_f has a retraction $r : Ef \to X$ over Y.

Remarks

1. This characterization is not "cofibrantly generated": we still don't know anything about the generating acyclic cofibrations.
2. Dually, projective cofibrations can sometimes* be characterized as retracts of their bar constructions. In 2-category theory these are called flexible objects, previously known to coincide with projectively cofibrant ones (Lack 2007).

* though not for simplicial sets

Semi-algebraic fibrations

Because we didn't say anything about the generating acyclic cofibrations, we can't use the same trick as before to prove the universe is fibrant. But we can do something else.

Definition

A semi-algebraic injective fibration is a map $f : X \to Y$ with

1. The property of being a pointwise fibration, and
2. The structure of a retraction for λ_f.

- Pullback preserves semi-algebraic injective fibration structures.
- f is an injective fibration iff it has *some* such structure.

Universes for injective fibrations

It is a fact that $[\mathscr{D}^{\mathrm{op}}, \mathscr{S}] \simeq [\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$ for some ordinary small category $\mathscr{C} = \mathbb{\Delta} \rtimes \mathscr{D}$.

Definition

Define $U \in [\mathscr{D}^{\mathrm{op}}, \mathscr{S}] \simeq [\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$, where for $c \in \mathscr{C}$, $U(c)$ is a set of κ-small semi-algebraic injective fibrations over $\mathbf{y}_c = \mathscr{C}(-, c)$. The functorial actions are by pullback.

Similarly, we define $\widetilde{U}$ using sectioned fibrations, and $\pi : \widetilde{U} \to U$.

The universal fibration is a fibration

Theorem

$\pi : \widetilde{U} \to U$ is a (semi-algebraic) injective fibration.

Proof.

We have $U = \operatorname{colim}_{x \in U(c)} \mathbf{y}_c$. Then $x^*\pi : x^*\widetilde{U} \to \mathbf{y}_c$ is a semi-algebraic injective fibration for all $x : \mathbf{y}_c \to U$, and pullback along any $\mathbf{y}_{c'} \to \mathbf{y}_c$ is compatible with these structures.

The functorial factorization E preserves pullbacks, and pullbacks preserve colimits, so $E\pi = \operatorname{colim}_{x \in U(c)} E_{x^*\pi}$, and the compatible retractions for $\lambda_{x^*\pi}$ induce a retraction for λ_π.
(Note π is a pointwise fibration, by the same representable-codomain arguments as before.) □

As before, this is the crux of the matter; the rest is straightforward.

Conclusion

Theorem

Every Grothendieck $(\infty, 1)$-topos can be presented by a model category that interprets homotopy type theory, with strict univalent universes closed under Σ-, Π-, and identity types.

Proof.

Present it as an accessible left exact localization of an injective model structure on simplicial presheaves $[\mathscr{D}^{\mathrm{op}}, \mathscr{S}]$. Then $[\mathscr{D}^{\mathrm{op}}, \mathscr{S}]$ has a universe, hence so does the localization. □