# Linear logic for constructive mathematics

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# Outline

1. Intuitionistic constructive mathematics
2. Linear constructive mathematics
3. The standard interpretation
4. The hidden linear nature of constructive mathematics

# Historical context

In the late 19th and early 20th centuries, two new trends in mathematics emerged in opposition:

- An increasing use of highly abstract concepts and non-constructive methods of proof (e.g. Cantorian set theory).
- A reaction insisting that proofs ought to remain constructive, associated with Kronecker, Poicaré, Weyl, and especially Brouwer and Heyting.

# A non-constructive proof

**Theorem**

*There exist irrational numbers $\alpha$ and $\beta$ such that $\alpha^\beta$ is rational.*

**Proof.**

Suppose for contradiction that $\alpha^\beta$ is irrational if $\alpha$ and $\beta$ are.

Taking $\alpha = \beta = \sqrt{2}$, we see $\sqrt{2}^{\sqrt{2}}$ is irrational.

Then taking $\alpha = \sqrt{2}^{\sqrt{2}}$ and $\beta = \sqrt{2}$, we get that $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = (\sqrt{2})^{\sqrt{2}\cdot\sqrt{2}} = (\sqrt{2})^2 = 2$ is irrational, a contradiction. □

The theorem claims that something exists, but the proof doesn't construct a particular such thing, so we are left with no idea exactly what the numbers $\alpha$ and $\beta$ are.

# Intuitionistic logic

To eliminate non-constructive proofs, Brouwer and Heyting formulated a new intuitionistic logic with the property that *all valid proofs are necessarily constructive*. Its features include:

- Proof by contradiction is not allowed. Hence a statement can be "not false" without being true: $\neg\neg P$ doesn't imply $P$.
- De Morgan's laws hold *except* $\neg(P \wedge Q) \rightarrow (\neg P \vee \neg Q)$.
- Similarly, $\neg\forall x.P(x)$ doesn't imply $\exists x.\neg P(x)$.
- The *law of excluded middle* $P \vee \neg P$ doesn't hold.

# The BHK interpretation

The Brouwer-Heyting-Kolmogorov (BHK) interpretation is an informal description of the meanings of intuitionistic connectives in terms of *what counts as a proof of them*.

- A proof of $P \wedge Q$ is a proof of $P$ and a proof of $Q$.
- A proof of $P \vee Q$ is a proof of $P$ or a proof of $Q$.
- A proof of $P \rightarrow Q$ is a construction transforming any proof of $P$ into a proof of $Q$.

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- A proof of $P \vee Q$ is a proof of $P$ or a proof of $Q$.
- A proof of $P \to Q$ is a construction transforming any proof of $P$ into a proof of $Q$.

Like the Tarskian definition of object-language truth in terms of meta-language truth, but involving constructions and proofs instead.

# BHK Negation

Brouwer defined $\neg P$ to be $P \to \bot$, i.e.

- A proof of $\neg P$ is a construction transforming any proof of $P$ into a proof of a contradiction.

This explains the properties of negation in intuitionistic logic:

- For an arbitrary $P$, we can't claim to have either a proof of $P$ or a construction transforming any proof of $P$ into a contradiction. (E.g. $P$ might be the Riemann hypothesis.) So $P \vee \neg P$ doesn't hold.
- If it would be contradictory to have a construction transforming any proof of $P$ into a contradiction, it doesn't follow that we have a proof of $P$. Hence $\neg\neg P$ doesn't imply $P$.

# Constructive analysis in intuitionistic logic

## Definition

A real number is an equivalence class of Cauchy sequences $x : \mathbb{N} \to \mathbb{Q}$, with

$$(x = y) \stackrel{\text{def}}{=} \forall \varepsilon > 0. \exists N. \forall n > N. |x_n - y_n| < \varepsilon.$$

## Problem

We expect the real numbers to be a "field", but $x \neq 0$ is not sufficient to define $\frac{1}{x}$.

$$(x \neq 0) \stackrel{\text{def}}{=} \neg \forall \varepsilon > 0. \exists N. \forall n > N. |x_n| < \varepsilon$$

which doesn't give us $\exists \varepsilon > 0$ with infinitely many $|x_n| > \varepsilon$, so we can't define a sequence $y : \mathbb{N} \to \mathbb{Q}$ to represent $\frac{1}{x}$.

# Apartness of reals

$$(x = y) \stackrel{\text{def}}{=} \forall \varepsilon > 0. \exists N. \forall n > N. |x_n - y_n| < \varepsilon.$$

**Definition**

Two real numbers $x, y$ are apart if

$$(x \# y) \stackrel{\text{def}}{=} \exists \varepsilon > 0. \forall N. \exists n > n. |x_n - y_n| \geq \varepsilon.$$

**Theorem**

*If $x \# 0$, then there exists $y$ with $xy = 1$.*

This is a more useful notion of “field”.

# Abstract apartness

**Definition**

An apartness relation on a set $A$ satisfies

1. $\neg(x \# x)$.
2. If $x \# y$, then $y \# x$.
3. If $x \# z$, then either $x \# y$ or $y \# z$.

Disequality $\neg(x = y)$ satisfies 1–2, but not generally 3.

**Definition**

An apartness group $G$ satisfies

- If $x \# y$, then $x^{-1} \# y^{-1}$.
- If $xu \# yv$, then either $x \# y$ or $u \# v$.

Similarly we have apartness rings, etc.

# Antisubgroups

### A new problem

If $H$ is a subgroup of an apartness group $G$, the quotient $G/H$ may no longer have an apartness.

### Definition

An antisubgroup is a subset $A \subseteq G$ of an apartness group with

- For all $x \in A$ we have $x \# e$.
- If $xy \in A$, then either $x \in A$ or $y \in A$.
- If $x \in A$ then $x^{-1} \in A$.

### Theorem

*If $A$ is an antisubgroup, then $G \setminus A$ is a subgroup and $G/(G \setminus A)$ is an apartness group.*

# And so on

- Anti-ideals, anti-subalgebras
- $x < y$ and $y \leq x$ are not each other's negations.
- Apartness spaces instead of topological spaces
- ...

*Experience shows that it is not necessary to define inequality in terms of negation. For those cases in which an inequality relation is needed, it is better to introduce it affirmatively.*
*– Errett Bishop, Foundations of constructive analysis*

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# An old joke

Patient: Doctor, it hurts when I do this.
Doctor: Then don't do that!

# An old joke

Patient: Doctor, it hurts when I do this.
Doctor: Then don't do that!

Constructivist: We define $\neg P$ to mean $P \to \bot$. But this definition is not really useful for much of anything.
Wag: Then don't define it like that!

# A better negation

A more useful notion of negation is the formal de Morgan dual.

$$
\begin{aligned}
\neg(P \vee Q) &\stackrel{\text{def}}{=} \neg P \wedge \neg Q \\
\neg(P \wedge Q) &\stackrel{\text{def}}{=} \neg P \vee \neg Q \\
\neg \exists x. P(x) &\stackrel{\text{def}}{=} \forall x. \neg P(x) \\
\neg \forall x. P(x) &\stackrel{\text{def}}{=} \exists x. \neg P(x)
\end{aligned}
$$

- A constructive proof of $\exists x. P(x)$ must *provide an example*.
- Similarly, a *constructive disproof* of $\forall x. P(x)$ should *provide a counterexample!*

# Constructive proof by contradiction?

This negation is involutive, $\neg\neg P = P$. Therefore, proof by contradiction is allowed. Huh?

What's nonconstructive about proof by contradiction? To prove $\exists x.P(x)$ by contradiction, we assume its negation $\forall x.\neg P(x)$. But in order to use this hypothesis at all, we have to *apply it to some* $x$! So it would seem that we *are* necessarily constructing something.

# Non-constructive proof by contradiction

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**Proof.**

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Non-constructivity enters if we use the contradiction hypothesis *more than once*, so that it's not clear *which* x is the example.

# Towards linear logic

*. . . take a proof of the existence or the disjunction property; we use the fact that the last rule used is an introduction, which we cannot do classically because of a possible contraction. Therefore, in the. . . intuitionistic case, $\vdash$ serves to mark a place where contraction. . . is forbidden. . . . Once we have recognized that the constructive features of intuitionistic logic come from the dumping of structural rules on a specific place in the sequents, we are ready to face the consequences of this remark: the limitation should be generalized to other rooms, i.e. weakening and contraction disappear.*

*– Jean-Yves Girard, "Linear Logic"*

# Constructivity through linear logic

- We divide the hypotheses into linear and nonlinear ones. The linear ones can only be used once in the course of a proof.
- All "hypotheses for contradiction" in a proof by contradiction are linear hypotheses.
- Similarly, $P \multimap Q$ is a linear implication that uses $P$ only once. Thus it is contraposable, $(P \multimap Q) = (\neg Q \multimap \neg P)$.

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- Actually, linearity is the "default" status. We mark the "nonlinear" hypotheses with a modality, $!P$.
- Technically this is affine logic: we only require "linear hypotheses" to be used at most once.

# The real numbers in linear logic

**Definition**

For real numbers defined by Cauchy sequences $x, y : \mathbb{N} \to \mathbb{Q}$,

$$(x = y) \stackrel{\text{def}}{=} \forall \varepsilon > 0. \exists N. \forall n > N. |x_n - y_n| < \varepsilon.$$

We then have

$$\begin{aligned}(x \neq y) &\stackrel{\text{def}}{=} \neg(x = y) \\ &= \exists \varepsilon > 0. \forall N. \exists n > N. |x_n - y_n| \geq \varepsilon.\end{aligned}$$

exactly the intuitionistic definition of $x \# y$.

**Theorem (in linear logic)**

*The real numbers are a field: if $x \neq 0$ then there is a $y$ with $xy = 1$.*

# The classical disjunction

- In classical logic, $(P \vee Q) = (\neg P \to Q) = (\neg Q \to P)$.
  This is no longer true in intuitionistic logic.
- It also fails in linear logic for the "constructive" $\vee$.
  But by contraposition, we do have $(\neg P \multimap Q) = (\neg Q \multimap P)$, defining another kind of disjunction that is weaker than $\vee$.

$$
\begin{aligned}
&(P \parr Q) && \text{“}P \text{ par } Q\text{”} \\
&\quad = (\neg P \multimap Q) && \text{“}P \text{ or else } Q\text{”} \\
&\quad = (\neg Q \multimap P) && \text{“}P \text{ unless } Q\text{”}.
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\end{aligned}
$$

- $\vee$-excluded middle $P \vee \neg P$ fails. But $\parr$-excluded middle $(P \parr \neg P) = (\neg P \multimap \neg P)$ is a tautology!
- $\vee$ supports proof by cases; $\parr$ supports the disjunctive syllogism.

# Inequality in linear logic

- Classically we have $(x \leq y) \leftrightarrow (x = y) \vee (x < y)$ and $(x \leq y) \vee (y \leq x)$ for real numbers $x, y$.
- Both fail intuitionistically and linearly, but linearly we do have $(x \leq y) \multimap\!\circ\, (x = y) \parr (x < y)$ and $(x \leq y) \parr (y \leq x)$.
- Pronounce $x \leq y$ as "$x$ is less than or else equal to $y$"?

# Additive and multiplicative

The de Morgan dual of $\parr$ is another conjunction, $(P \otimes Q) = \neg(\neg P \parr \neg Q)$, which allows us to use $P$ and $Q$ once each (instead of once in total, like $P \wedge Q$).

- $\wedge$ and $\vee$ are called additive.
- $\otimes$ and $\parr$ are called multiplicative.
- $\otimes$ and $\vee$ are called positive.
- $\wedge$ and $\parr$ are called negative.

# Outline

# A meaning explanation for affine logic

That's all well and good, but what does this wacky logic mean?

The reason the BHK interpretation gives a non-involutive negation is that it privileges proofs over refutations. We can instead give a meaning interpretation that treats them on an equal footing.

- A proof of $P \wedge Q$ is a proof of $P$ and a proof of $Q$.
  A refutation of $P \wedge Q$ is a refutation of $P$ or a refutation of $Q$.
- A proof of $P \vee Q$ is a proof of $P$ or a proof of $Q$. A refutation of $P \vee Q$ is a refutation of $P$ and a refutation of $Q$.
- A proof of $\neg P$ is a refutation of $P$.
  A refutation of $\neg P$ is a proof of $P$.

# The meaning of the multiplicatives

- A proof of $P \parr Q$ is a construction transforming any refutation of $P$ into a proof of $Q$, and a construction transforming any refutation of $Q$ into a proof of $P$. A refutation of $P \parr Q$ is a refutation of $P$ and a refutation of $Q$.
- A proof of $P \multimap Q$ is a construction transforming any proof of $P$ into a proof of $Q$, and a construction transforming any refutation of $Q$ into a refutation of $P$. A refutation of $P \multimap Q$ is a proof of $P$ and a refutation of $Q$.

Note:

- $P \vee Q$ and $P \parr Q$ have the same refutations, different proofs.
- $P \wedge Q$ and $P \otimes Q$ (not shown) have the same proofs, different refutations.

# Towards a formalization

Like the BHK interpretation, this meaning explanation is informal, and nonspecific about what a “construction” is.

But the relationship between the two meaning explanations can be made formal: we interpret each linear proposition $P$ as a *pair* of intuitionistic propositions $(P^+, P^-)$ representing its proofs and refutations respectively.

We can make this precise using *algebraic semantics*.

# Heyting algebras

### Definition

A Heyting algebra is a cartesian closed lattice, i.e. a poset $H$ with

- A top element $\top$ and bottom element $\bot$.
- Meets $P \wedge Q$ and joins $P \vee Q$.
- An "implication" with $(P \wedge Q) \leq R$ iff $P \leq (Q \to R)$.

Heyting algebras are the algebraic semantics of intuitionistic logic, just like Boolean algebras are for classical logic.

# $*$-autonomous lattices

## Definition

A semicartesian $*$-autonomous lattice is a poset $L$ with

- A top element $\top$ and bottom element $\bot$.
- Meets $P \wedge Q$ and joins $P \vee Q$.
- An associative tensor product $\otimes$ with unit $\top$.
- An involution $\neg$ such that $(P \otimes Q) \leq \neg R$ iff $P \leq \neg(Q \otimes R)$.

Define $(P \parr Q) = \neg(\neg P \otimes \neg Q)$ and $(P \multimap Q) = (\neg P \parr Q)$.

Semicartesian $*$-autonomous lattices* are the algebraic semantics of affine logic.

* with a Seely comonad

# A Chu construction

## Theorem

*For any Heyting algebra $H$, there is a semicartesian $*$-autonomous lattice defined by:*

- *Its elements are pairs $P = (P^+, P^-)$ where $P^+, P^- \in H$ with $P^+ \wedge P^- = \bot$. (Think $P^+ =$ proofs, $P^- =$ refutations.)*
- *We define $P \leq Q$ to mean $P^+ \leq Q^+$ and $Q^- \leq P^-$.*
- $\top = (\top, \bot)$ *and* $\bot = (\bot, \top)$
- $P \wedge Q = (P^+ \wedge Q^+, P^- \vee Q^-)$.
- $P \vee Q = (P^+ \vee Q^+, P^- \wedge Q^-)$.
- $P \otimes Q = (P^+ \wedge Q^+, (P^+ \to Q^-) \wedge (Q^+ \to P^-))$
- $P \parr Q = ((P^- \to Q^+) \wedge (Q^- \to P^+), P^- \wedge Q^-)$
- $P \multimap Q = ((P^+ \to Q^+) \wedge (Q^- \to P^-), P^+ \wedge Q^-)$

# The standard interpretation

The Chu construction is a much more general operation that builds an $*$-autonomous *category* from any closed symmetric monoidal category with any chosen object (replacing $\bot$).

Our special case of a Heyting algebra $H$ with bottom element $\bot$ yields a translation of affine propositional logic into intuitionistic propositional logic. It can also be extended to first-order logic:

$$\exists x.P(x) = (\exists x.P^+(x), \forall x.P^-(x))$$
$$\forall x.P(x) = (\forall x.P^+(x), \exists x.P^-(x)).$$

We call this the standard interpretation.

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1. Intuitionistic constructive mathematics
2. Linear constructive mathematics
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4. The hidden linear nature of constructive mathematics

# Linear vs intuitionistic logic

Linear logic was *originally conceived* by Girard as a constructive logic with an involutive negation.

> *. . . the linear negation . . . is a constructive and involutive negation; by the way, linear logic works in a classical framework, while being more constructive than intuitionistic logic.*
>
> *– Jean-Yves Girard, "Linear logic", 1987*

Yet, in the 40 years since, essentially no constructive mathematicians have adopted linear logic as a replacement for intuitionistic logic.

# Why not?

Why not? I can only speculate, but some reasons might include:

1. They don't know about linear logic.
2. They think it's just a weird thing for proof theorists.
3. They think it's only about *feasible* computation.
   (It can be about that, but only by restricting the rules for !)
4. They don't understand the meaning of the connectives.
5. They can't figure out when to use $\otimes/\parr$ versus $\wedge/\vee$.
6. There's no "migration path" from intuitionistic logic.
7. It doesn't "do anything" for them that intuitionistic logic doesn't.

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5. They can't figure out when to use $\otimes/\parr$ versus $\wedge/\vee$.
6. There's no "migration path" from intuitionistic logic.
7. It doesn't "do anything" for them that intuitionistic logic doesn't.

The standard interpretation can help!

# An empirical observation

**Fact**

Many definitions in intuitionistic constructive mathematics (including some of the oddest-looking ones) arise naturally by

1. writing a classical definition in linear logic (making choices between $\otimes/\parr$ and $\wedge/\vee$).
2. passing across the standard interpretation.

*"Constructive mathematicians have been using linear logic without realizing it!"*

# Example 1: Apartness

$$(x = y)^+ = (x = y)$$
$$(x = y)^- = (x \# y)$$

| Linear logic | Intuitionistic logic |
|---|---|
| Relation $x = y$ | Relations $x = y$ and $x \# y$ with $\neg((x = y) \wedge (x \# y))$ |
| $x = x$ | $x = x$ and $\neg(x \# x)$ |
| $(x = y) \multimap (y = x)$ | $(x = y) \to (y = x)$<br>$(x \# y) \to (y \# x)$ |
| $(x = y) \wedge (y = z) \multimap (x = z)$ | $(x = y) \wedge (y = z) \to (x = z)$<br>$(x \# z) \to (x \# y) \vee (y \# z)$ |
| equality | equality + apartness |

# Example 2: Order

| Linear logic | Intuitionistic logic |
|---|---|
| Relation $x \leq y$ | Relations $x \leq y$ and $y < x$ with $\neg((x \leq y) \wedge (y < x))$ |
| $x \leq x$ | $x \leq x$ and $\neg(x < x)$ |
| $(x \leq y) \wedge (y \leq z) \multimap (x \leq z)$ | $(x \leq y) \wedge (y \leq z) \to (x \leq z)$<br>$(z < x) \to (z < y) \vee (y < z)$ |
| $(x \leq y) \wedge (y \leq x) \multimap (x = y)$ | $(x \leq y) \wedge (y \leq x) \to (x = y)$<br>$(x \# y) \to (x < y) \vee (y < x)$ |
| $(x \leq y) \vee (y \leq x)$ | $(x \leq y) \vee (y \leq x)$ |
| $(x \leq y) \parr (y \leq x)$ | $(x < y) \to (x \leq y)$ |
| partial order | strict + non-strict order pair |

# Example 3: Sets and functions

| Linear logic | Intuitionistic logic |
|---|---|
| Subset $U \subseteq A$ | Subsets $U, \not{U} \subseteq A$ with<br>$(x \in U) \wedge (y \in \not{U}) \to (x \# y)$<br>(a *complemented subset*) |
| $(x = y) \wedge (x \in U) \multimap (y \in U)$ | $(x = y) \wedge (x \in U) \to (y \in U)$<br>$(y \in \not{U}) \to (x \# y) \vee (x \in \not{U})$ |
| $U \neq \emptyset$ | $\exists x.(x \in U)$<br>($U$ is *inhabited*) |
| Function $f : A \to B$<br>$(x = y) \multimap (f(x) = f(y))$ | Function $f : A \to B$<br>$(x = y) \to (f(x) = f(y))$<br>$(f(x) \# f(y)) \to (x \# y)$<br>($f$ is *strongly extensional*) |

# Example 4: Algebra

| Linear logic | Intuitionistic logic |
|---|---|
| Group $G$ | Group $G$ with apartness |
| $(x = y) \wedge (u = v) \multimap (xu^{-1} = yv^{-1})$ | $(xu^{-1} \# yv^{-1}) \to (x \# y) \vee (u \# v)$ |
| Subgroup $H$ | Subgroup $H$, antisubgroup $\not{H}$ |
| $x \in H \wedge y \in H \multimap xy \in H$ | $x \in H \wedge y \in H \to xy \in H$<br>$xy \in \not{H} \to x \in \not{H} \vee y \in \not{H}$ |
| Ring | Ring with apartness |
| Ideal | Ideal + anti-ideal |
| ⋮ | ⋮ |

# Example 5: Topology

| Linear logic | Intuitionistic logic |
|---|---|
| Topological space $X$ as closure operator | Space $X$ with topology and* point-set apartness $x \bowtie U$ |
| $U \subseteq \mathrm{cl}(U)$ | $U \subseteq \mathrm{cl}(U)$<br>$(x \bowtie U) \to (x \notin U)$ |
| $\mathrm{cl}(\emptyset) = \emptyset$ | $\mathrm{cl}(\emptyset) = \emptyset$ and $x \bowtie \emptyset$ |
| $(x \in \mathrm{cl}(U \cup V))$<br>$\multimap (x \in \mathrm{cl}(U)) \parr (x \in \mathrm{cl}(V))$ | $(x \in \mathrm{cl}(U \cup V)) \wedge (x \bowtie U)$<br>$\to (x \in \mathrm{cl}(V))$<br>$(x \bowtie U) \wedge (x \bowtie V)$<br>$\to (x \bowtie (U \cup V))$ |

Classically, $\mathrm{cl}(U \cup V) = \mathrm{cl}(U) \cup \mathrm{cl}(V)$, but not intutitionistically.
The standard interpretation yields the correct substitute(s).

* some details being fudged here

# The linear nature of constructive mathematics

All the definitions appearing in the right-hand columns:

1. Were defined and studied by constructive mathematicians for purely practical reasons.
2. Look weird and backwards to a classical mathematician.
3. Require “backwards” bookkeeping that is easy to get wrong.
4. Arise automatically from the standard interpretation.
5. Are automatically “kept track of” by working in linear logic.

# The point of the standard interpretation

1. Explains some of the proliferation of constructive concepts in terms of the choices between $\otimes/\parr$ and $\wedge/\vee$.
2. Instead of $\neg\neg(P \vee Q)$, the "classical disjunction" is $P \parr Q$, which has more constructive content.
3. Linear logic can be a "higher-level" tool on top of intuitionistic logic, to automatically handle apartness bookkeeping.
4. A new way to "constructivize" classical concepts, by writing them in linear logic and applying the standard interpretation.
5. Solves some (minor) open problems in intuitionistic constructive mathematics, such as giving:
    - The correct "union axiom" for a closure space.
    - A notion of "metric space" that includes Hausdorff metrics.

Thanks for listening!

*Linear logic for constructive mathematics*
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